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Two-scale convergence in Sobolev spaces for a two-dimensional case

Hội tụ two-scale trong các không gian Sobolev cho một trường hợp hai chiều

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Abstract

We present some properties of the two-scale convergence in Sobolev spaces for a two-dimensional case.

Keywords: Two-scale homogenization; weak-two scale convergence; two-scale convergence in Sobolev spaces; two-dimensional

Tóm tắt

Trong bài báo này, chúng tôi trình bày một số tính chất của hội tụ two-scale trong các không gian Sobolev cho một trường hợp hai chiều.

Từ khóa: Đồng nhất hóa two-scale; hội tụ two-scale yếu; hội tụ two-scale trong các không gian Sobolev; hai chiều

1. Introduction

Consider a variable $\boldsymbol{x} = (x^1, x^2)$ and a bounded reference domain Ω in dimension two, where Ω is defined as $\Omega^1 \times \Omega^2 \in \mathbb{R} \times \mathbb{R}$. When the conventional weak limit cannot be used in the context of the two-scale homogenization theory, the two-scale limit [1], which Nguetseng developed in 1989, may be used instead. Keeping this in mind, we first give a brief overview of the usual weak convergence and weak two-scale convergence before presenting some properties of two-scale convergence in Sobolev spaces [2, 3, 4, 5], for the case of two dimensions.

2. Preliminaries

The collection $\{1, 2\}$ contains Latin indices. Functions are represented by italic capitals (e.g., f), vector fields in $\mathbb{R}^2$ and 2×2 matrix fields over Ω are symbolized by bolds letters (e.g., $\boldsymbol{v}$ and $\boldsymbol{T}$). Italic capital letters (e.g., $L^2(\Omega)$), boldface Roman capital letters (e.g., $\boldsymbol{L}$), and special Roman capital letters (e.g., $\mathbb{L}$) are used to designate the space of functions, vector fields, and 2×2 matrix fields defined over $\Omega = \mathbb{R}^2$, respectively.

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We employ the following list of notations throughout the study [2]:

- $Y := [0,1]^2$ denotes the reference periodic cell.
- $\boldsymbol{C}_0(\Omega)$ stands for the space of functions that vanish at infinity.
- $\boldsymbol{C}^{\infty}_{\text{per}}(Y)$ represents the Y-periodic $\boldsymbol{C}^{\infty}$ vector-valued functions in $\mathbb{R}^2$. Herein, Y-periodic implies 1-periodic in each variable $y^i, i = 1,2$.
- $\boldsymbol{H}^1_{\text{per}}(Y)$, being the closure for the $\boldsymbol{H}^1$-norm of $\boldsymbol{C}^{\infty}_{\text{per}}(Y)$, describes the space of vector-valued functions $\boldsymbol{v} \in \boldsymbol{L}^2(Y)$ such that $\boldsymbol{v}(y)$ is Y-periodic in $\mathbb{R}^2$.
- The mean value of function $\boldsymbol{v}(y)$ is

$$\langle \boldsymbol{v} \rangle_y = \frac{1}{|Y|} \int_Y \boldsymbol{v}(y)\, \mathrm{d}y,$$

where $|Y|$ denotes the volume of Y.

- $$\boldsymbol{H}_{\text{per}}(Y) := \{\boldsymbol{v} \in \boldsymbol{H}^1_{\text{per}}(Y) \,|\, \langle \boldsymbol{v} \rangle_y = 0\}.$$
- The $\cdot$ represents the canonical inner products in $\mathbb{R}^2$ and $\mathbb{R}^{2\times 2}$.

The form of Sobolev norm $\|\cdot\|_{\boldsymbol{W}^{1,2}_0(\Omega)}$ is

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,2}_0(\Omega)} = (\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)})^{\frac{1}{2}};$$

with $\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} := \||\boldsymbol{v}|\|_{L^2(\Omega)}$, in which $|\boldsymbol{v}|$ is the Euclidean norm of the 2-component vector-valued function $\boldsymbol{v}$, and $\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2(\Omega)} := \||\nabla \boldsymbol{v}|\|_{L^2(\Omega)}$, where $|\nabla \boldsymbol{v}|$ is the Frobenius norm of the 2×2 matrix $\nabla \boldsymbol{v}$. Recall that the Frobenius norm on $\mathbb{L}^2(\Omega)$ is expressed by $|\boldsymbol{X}|^2 := \boldsymbol{X}\cdot\boldsymbol{X} = \text{tr}(\boldsymbol{X}^{\text{T}}\boldsymbol{X})$.

We let ϵ be a natural small scale. Following [6, 7, 8, 9], we investigate $\boldsymbol{u}_\epsilon(\boldsymbol{x}) \in \boldsymbol{W}^{1,2}_0(\Omega)$ depending only on x^1 in the form $\boldsymbol{u}_\epsilon(\boldsymbol{x}) = \boldsymbol{u}_\epsilon(x^1)$, with Neumann type boundary conditions. Thanks to [10], we do not distinguish between a function on $\mathbb{R}$ and its extension to $\mathbb{R}^2$ as a function of the first variable alone. Assume that $\boldsymbol{u}_\epsilon(x^1) = \boldsymbol{u}\left(\frac{x^1}{\epsilon}\right)$ is a periodic function in x^1 having period ϵ, equivalently, $\boldsymbol{u}\left(\frac{x^1}{\epsilon}\right) = \boldsymbol{u}(y^1)$ is a periodic function in y^1 possesing period 1. It implies that for any integer k,

$$\boldsymbol{u}_\epsilon(x^1) = \boldsymbol{u}_\epsilon(x^1+\epsilon) = \boldsymbol{u}_\epsilon(x^1+k\epsilon),$$

that is,

$$\boldsymbol{u}\left(\frac{x^1}{\epsilon}\right) = \boldsymbol{u}\left(\frac{x^1}{\epsilon}+1\right) = \boldsymbol{u}\left(\frac{x^1}{\epsilon}+k1\right) = \boldsymbol{u}(y^1+k).$$

To show the key concept, we focus on the following case from strain-limiting elasticity [11, 12, 13]:

$$-\text{div}(\boldsymbol{\kappa}(x^1, |\boldsymbol{D}\boldsymbol{u}_\epsilon|)\boldsymbol{D}\boldsymbol{u}_\epsilon) = \boldsymbol{f} \text{ in } \Omega, \boldsymbol{u}_\epsilon = \boldsymbol{0} \text{ on } \partial\Omega, \quad (1)$$

where $\boldsymbol{D}\boldsymbol{u}_\epsilon$ stands for the classical linearized strain tensor

$$\boldsymbol{D}\boldsymbol{u}_\epsilon = \frac{1}{2}(\nabla \boldsymbol{u}_\epsilon + \nabla \boldsymbol{u}_\epsilon^{\text{T}}).$$

An equivalent form of (1) is

$$-\text{div}(\boldsymbol{a}(x^1, \boldsymbol{D}\boldsymbol{u}_\epsilon)) = \boldsymbol{f} \text{ in } \Omega, \boldsymbol{u}_\epsilon = \boldsymbol{0} \text{ on } \partial\Omega, \quad (2)$$

where $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$,

$$\boldsymbol{a}(x^1, \boldsymbol{D}\boldsymbol{u}_\epsilon) = \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\boldsymbol{u}_\epsilon|)\boldsymbol{D}\boldsymbol{u}_\epsilon = \frac{\boldsymbol{D}\boldsymbol{u}_\epsilon}{1-\beta_\epsilon(x^1)|\boldsymbol{D}\boldsymbol{u}_\epsilon|}$$

is a high-contrast coefficient $\boldsymbol{a}(x^1,\cdot)$ and assumed to be grately heterogeneous with regard to $\boldsymbol{x} = (x^1, x^2)$, and $\boldsymbol{f} \in \boldsymbol{H}^1_*(\Omega) \subset \boldsymbol{L}^2(\Omega) \subsetneq \boldsymbol{H}^{-1}(\Omega)$ is an external force.

Let

$$\mathcal{Z} := \left\{\boldsymbol{\zeta} \in \mathbb{L}^2(\Omega) \,|\, 0 \le |\boldsymbol{\zeta}| < \frac{1}{\beta_\epsilon(x^1)} < 1\right\}. \quad (3)$$

3. Weak convergence

We review the basic notions of the theory of two-scale convergence [4, 5]. Two-scale convergence here can be thought as a generalized version of the traditional weak convergence in the Hilbert space $\boldsymbol{L}^2(\Omega)$, which is described below [4].

Consider a sequence of functions $\boldsymbol{u}_\epsilon \in \boldsymbol{L}^2(\Omega)$. By definition, $(\boldsymbol{u}_\epsilon)$ is bounded in $\boldsymbol{L}^2(\Omega)$ if

$$\limsup_{\epsilon\to 0}\int_\Omega |\boldsymbol{u}_\epsilon|^2\,\mathrm{d}x \le c < \infty,$$

with some positive constant c.

One states that a sequence $(\boldsymbol{u}_\epsilon(\boldsymbol{x})) \in \boldsymbol{L}^2(\Omega)$ is weakly convergent to $\boldsymbol{u}(\boldsymbol{x}) \in \boldsymbol{L}^2(\Omega)$ as $\epsilon \to 0$, abbreviated by $\boldsymbol{u}_\epsilon \rightharpoonup \boldsymbol{u}$, if for any test function $\boldsymbol{\phi} \in \boldsymbol{L}^2(\Omega)$,

$$\lim_{\epsilon\to 0}\int_\Omega \boldsymbol{u}_\epsilon(\boldsymbol{x})\cdot\boldsymbol{\phi}\,\mathrm{d}x = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{\phi}\,\mathrm{d}x. \tag{4}$$

Furthermore, a sequence $(\boldsymbol{u}_\epsilon)$ in $\boldsymbol{L}^2(\Omega)$ is determined to be strongly convergent to $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$ when $\epsilon \to 0$, represented by $\boldsymbol{u}_\epsilon \to \boldsymbol{u}$ if

$$\lim_{\epsilon\to 0}\int_\Omega \boldsymbol{u}_\epsilon\cdot\boldsymbol{v}_\epsilon\,\mathrm{d}x = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}x, \tag{5}$$

for any sequence $(\boldsymbol{v}_\epsilon) \in \boldsymbol{L}^2(\Omega)$ that is weakly convergent to $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$.

Over this paper, we let $Y = [0,1]^2$ be the cell of periodicity. (In our case, a periodic cell possesses the form $Y = [0,1] \times [0,1]$.) The mean value of a 1-periodic function $\boldsymbol{\psi}(y^1)$ is written as $\langle\boldsymbol{\psi}\rangle$:

$$\langle\boldsymbol{\psi}\rangle \equiv \int_{Y^1} \boldsymbol{\psi}(y^1)\,\mathrm{d}y^1,$$

where $Y^1 = [0,1]$, and $y^1 = \epsilon^{-1}x^1$.

Also, the notation $\boldsymbol{L}^2(Y)$ holds here not only for functions over Y but also for the space of functions in $\boldsymbol{L}^2(Y)$ extended by 1-periodicity to entire $\mathbb{R}^2$. In a similar way, $\boldsymbol{C}^\infty_{\text{per}}(Y)$ represents the space of infinitely differentiable 1-periodic functions over all $\mathbb{R}^2$.

4. Weak two-scale convergence

We recall the following definition of weak two-scale convergence in $L^2(\Omega)$ [2, 3, 4].

Definition 4.1. *Provided a bounded sequence* (u_ϵ) *in* $L^2(\Omega)$. *If there is some subsequence, still represented by* u_ϵ *and a function* $u(\boldsymbol{x}, y^1) \in L^2(\Omega\times Y^1)$ *(where* $Y^1 = [0,1]$*) such that*

$$\lim_{\epsilon\to 0}\int_\Omega u_\epsilon(\boldsymbol{x})\left(\phi(\boldsymbol{x})h\left(\frac{x^1}{\epsilon}\right)\right)\mathrm{d}x = \int_{\Omega\times Y^1} u(\boldsymbol{x},y^1)(\phi(\boldsymbol{x})h(y^1))\,\mathrm{d}x\,\mathrm{d}y^1 \tag{6}$$

for any $h \in C^\infty_{per}(Y^1)$ *and any* $\phi \in C^\infty_0(\Omega)$, *then such a sequence* u_ϵ *is called weakly two-scale converge to* $u(\boldsymbol{x}, y^1)$. *This convergence is symbolized by* $u_\epsilon(\boldsymbol{x}) \rightharpoonup\!\!\rightharpoonup u(\boldsymbol{x}, y^1)$.

For vector (or matrix) $\boldsymbol{u}_\epsilon$, equation (6) leads to

$$\lim_{\epsilon\to 0}\int_\Omega \boldsymbol{u}_\epsilon(\boldsymbol{x})\cdot\boldsymbol{\Phi}\left(\boldsymbol{x},\frac{x^1}{\epsilon}\right)\mathrm{d}x = \int_{\Omega\times Y^1}\boldsymbol{u}(\boldsymbol{x},y^1)\cdot\boldsymbol{\Phi}(\boldsymbol{x},y^1)\,\mathrm{d}x\,\mathrm{d}y^1, \tag{7}$$

with any $\boldsymbol{\Phi} \in \boldsymbol{L}^2(\Omega; \boldsymbol{C}_{\text{per}}(Y^1))$, whose choice can be found in [14] (p. 8).

Remark 4.2. *For the class of test functions* $\phi \in C^\infty_0(\Omega), h \in C^\infty_{per}(Y^1)$ *in (6)'s condition, it can be extended (utilizing the density argument) to the class of test functions* $\phi \in C^\infty_0(\Omega), h \in L^2(Y^1)$.

Therefore, the convergence $u_\epsilon \rightharpoonup\!\!\rightharpoonup u$ means the convergence

$$u_\epsilon(\boldsymbol{x})b\left(\frac{x^1}{\epsilon}\right) \rightharpoonup\!\!\rightharpoonup u(\boldsymbol{x},y^1)b(y^1), \quad \forall\, b \in L^\infty(Y^1). \tag{8}$$

5. Two-scale convergence in Sobolev spaces

This section and its notation follow [4, 5]. Recall that a matrix (or vector) $\boldsymbol{Z} \in \mathbb{L}^1(\Omega)$ is referred to as *solenoidal* (writing $\operatorname{div}\boldsymbol{Z} = \operatorname{div}_x\boldsymbol{Z} = \boldsymbol{0}$) if

$$\int_\Omega \boldsymbol{Z}\cdot\boldsymbol{D}\boldsymbol{v}\,dx = 0 \quad \forall\boldsymbol{v}\in\boldsymbol{C}^\infty_0(\Omega).$$

Also, a 1-periodic matrix $\boldsymbol{Z} = \boldsymbol{Z}(y^1) \in \mathbb{L}^1(Y^1)$ is named *solenoidal* (writing $\operatorname{div}_y\boldsymbol{Z} = \boldsymbol{0}$) if

$$\int_{Y^1}\boldsymbol{Z}\cdot\boldsymbol{D}\boldsymbol{v}\,dy^1 = 0 \quad \forall\boldsymbol{v} = \boldsymbol{v}(y^1)\in\boldsymbol{C}^\infty_{\text{per}}(Y^1).$$

Let us discuss a few significant functional spaces.

The space $\boldsymbol{H}^1_{\text{per}}(Y^1)$ is the closure of $\boldsymbol{C}^\infty_{\text{per}}(Y^1)$ in $\boldsymbol{L}^2(Y^1)$ with regard to the norm

$$\|\boldsymbol{v}\|^2_{1,Y^1} = \int_{Y^1}(|\boldsymbol{v}|^2 + |\boldsymbol{D}\boldsymbol{v}|^2)\,dy^1.$$

The Poincaré inequality holds for the elements of this space, $\forall \boldsymbol{v} \in \boldsymbol{H}^1_{\text{per}}(Y^1)$ and $\int_{Y^1} \boldsymbol{v}\, dy^1 = \boldsymbol{0}$:

$$\int_{Y^1} |\boldsymbol{v}|^2\, dy^1 \le c \int_{Y^1} |\boldsymbol{D}\boldsymbol{v}|^2\, dy^1. \qquad (9)$$

Thus, on the subspace of $\boldsymbol{H}^1_{\text{per}}(Y^1)$ containing vector functions with zero mean value, the above norm has the equivalent form

$$\left(\int_{Y^1} |\boldsymbol{D}\boldsymbol{v}|^2\, dy^1\right)^{1/2},$$

and this subspace is comparable to the space of potential matrices (such as classical linearized strains in elasticity):

$$\mathbb{V}^2_{\text{pot}}(Y^1) = \{\boldsymbol{D}\boldsymbol{v} : \boldsymbol{v} \in \boldsymbol{H}^1_{\text{per}}(Y^1)\}. \qquad (10)$$

More precisely, $\boldsymbol{D}\boldsymbol{v} \in \mathscr{Z}$ as in (3), and we still use $\mathbb{V}^2_{\text{pot}}(Y^1)$ to include this given hypothesis in our paper.

Without confusion of notation, we can define the periodic Sobolev space $\boldsymbol{H}^1_{\text{per}}(Y^1)$ as the closure of $(\boldsymbol{u}, \boldsymbol{D}\boldsymbol{u})$, where $\boldsymbol{u} \in \boldsymbol{C}^\infty_{\text{per}}(Y^1)$, in $\boldsymbol{L}^2(Y^1) \times \mathbb{L}^2(Y^1)$. The elements of $\boldsymbol{H}^1_{\text{per}}(Y^1)$ are thus pairs $\bar{\boldsymbol{u}} = (\boldsymbol{u}, \boldsymbol{z})$, where the second component $\boldsymbol{z}$ is said to be the classical linearized strain tensor (the symmetric part of the gradient of the first component $\boldsymbol{u}$) and is denoted by $\boldsymbol{D}\boldsymbol{u}$.

It is guaranteed that $\mathbb{V}^2_{\text{pot}}(Y^1)$ is a closed subspace in $\mathbb{L}^2(Y^1)$ by the Poincaré inequality, and each of its elements can be represented by $\boldsymbol{D}\boldsymbol{u}$ with $\langle \boldsymbol{u} \rangle = \boldsymbol{0}$ in a unique way. By Theorem 4.7 in [15], every norm-closed subspace of $\mathbb{L}^2(Y^1)$ is the annihilator of its annihilator, so we have

$$\mathbb{V}^2_{\text{pot}} = (\mathbb{V}^2_{\text{sol}})^\perp \text{ and } \mathbb{V}^2_{\text{sol}} = (\mathbb{V}^2_{\text{pot}})^\perp. \qquad (11)$$

Thus, the following orthogonal decomposition of $\mathbb{L}^2(Y^1)$ holds:

$$\mathbb{L}^2(Y^1) = \mathbb{V}^2_{\text{pot}}(Y^1) \oplus \mathbb{V}^2_{\text{sol}}(Y^1), \qquad (12)$$

where $\mathbb{V}^2_{\text{sol}}(Y^1)$ is the collection of all solenoidal (1-periodic) matrices in $\mathbb{L}^2(Y^1)$.

Recall that we do not discriminate a function on Y^1 from its extension to Y as a function of the first variable only.

According to [16], the gradient's non-uniqueness is not really a trouble when determining an elliptic equation's solution. The pair $(\boldsymbol{u}, \boldsymbol{D}\boldsymbol{u})$ represents the given equation (1)'s solution, and its existence and uniqueness are inferred from the general theory of monotone operators. There are two factors that make a solution in this case unique: only one function in $\boldsymbol{H}^1_{\text{per}}$ and one of its gradients can make the equation satisfied.

We write $b = \operatorname{div} \boldsymbol{a}$ in order to demonstrate that there are $b \in L^1_{\text{per}}(Y^1)$ and vector-valued function $\boldsymbol{a} \in \boldsymbol{L}^1_{\text{per}}(Y^1)$ such that

$$\int_{Y^1} b\phi\, dy^1 = -\int_{Y^1} \boldsymbol{a} \cdot \boldsymbol{D}\phi\, dy^1 \quad \forall \phi \in C^\infty_{\text{per}}(Y^1). \qquad (13)$$

Equivalently,

$$\int_{\mathbb{R}} b\phi\, dy^1 = -\int_{\mathbb{R}} \boldsymbol{a} \cdot \boldsymbol{D}\phi\, dy^1, \quad \forall \phi \in C^\infty_0(\mathbb{R}). \qquad (14)$$

Choosing $\phi = 1$ in (14), we deduce that each function b accepting the expression $b = \operatorname{div} \boldsymbol{a}$ posses a mean value of zero:

$$\int_{Y^1} b\, dy^1 = 0.$$

The following theorem is based on [5], as does its proof.

Theorem 5.1. *With $\boldsymbol{a} \in \boldsymbol{L}^2(Y^1)$, the collection of functions $b \in L^2(Y^1)$, denoted by $b = div\, \boldsymbol{a}$, is dense in the subspace of functions in $L^2(Y^1)$ having mean value 0.*

Proof. Let B represent the collection of functions $b \in L^2(Y^1)$ that is denoted by $b = \operatorname{div} \boldsymbol{a}, \boldsymbol{a} \in \boldsymbol{L}^2(Y^1)$. The annihilator $B^\perp$ is defined as the collection of functions $k \in L^2(Y^1)$ such that $\int_{Y^1} k\, dy^1 = 0$ and $\int_{Y^1} kb\, dy^1 = 0$ for any $b \in B$.

Recall that $B^\perp = \{b^* \in (L^2(Y^1))^* = L^2(Y^1) | \langle b^*, b\rangle = 0 \quad \forall b \in B\}$. If we can demonstrate that $B^\perp = \{0\}$, then it follows that B is dense in $L^2(Y^1)$ by invoking Theorem 4.7 in [15] (saying $(B^\perp)^\perp$ is the norm-closure of B in $L^2(Y^1)$).

Fixing $k \in B^\perp$, we investigate the periodic problem: Find $(u, \boldsymbol{D}u) \in H^1_{\text{per}}(Y^1)$ such that

$$\int_{Y^1} (\boldsymbol{D}u \cdot \boldsymbol{D}\phi + u\phi)\, dy^1 = \int_{Y^1} k\phi\, dy^1, \qquad (15)$$

for any $\phi \in C^\infty_{\text{per}}(Y^1)$ (being test function, it can be chosen to be $(\phi, \boldsymbol{D}\phi) \in H^1_{\text{per}}(Y^1)$). It is well known that this issue can be solved. The obvious result is that $k - u \in B$, and since $k \in B^\perp$, we get $0 = \int_{Y^1} k(k-u)\,dy^1 = \int_{Y^1}(|k|^2 - uk)\,dy^1$. Applying the Hölder inequality, we obtain

$$\int_{Y^1} |k|^2\,dy^1 = \int_{Y^1} uk\,dy^1 \le \left(\int_{Y^1} |u|^2 dy^1\right)^{1/2}\left(\int_{Y^1} |k|^2 dy^1\right)^{1/2},$$

that is,

$$\int_{Y^1} |k|^2\,dy^1 \le \int_{Y^1} |u|^2\,dy^1. \tag{16}$$

By letting $\phi = u$ in (15), we have

$$\int_{Y^1} (|\boldsymbol{D}u|^2 + |u|^2)\,dy^1 = \int_{Y^1} ku\,dy^1 \le \left(\int_{Y^1} |k|^2\,dy^1\right)^{1/2}\left(\int_{Y^1} |u|^2\,dy^1\right)^{1/2}.$$

Therefore,

$$\int_{Y^1} |u|^2\,dy^1 \le \int_{Y^1} |k|^2\,dy^1$$

and

$$\int_{Y^1} (|\boldsymbol{D}u|^2 + |u|^2)\,dy^1 \le \int_{Y^1} |k|^2\,dy^1.$$

This along with (16) yields

$$\int_{Y^1} |u|^2\,dy^1 = \int_{Y^1} |k|^2\,dy^1, \int_{Y^1} |\boldsymbol{D}u|^2\,dy^1 = 0.$$

It holds by the later result that u is constant for a.e. $y^1 \in Y^1$. This information and equation (15) imply that k is constant for a.e. $y^1 \in Y^1$. Since $\int_{Y^1} k\,dy^1 = 0$, it follows that $k = 0$. The proof is thus finished.

The following theorem and its proof are derived from [5].

Theorem 5.2. *Let $(\boldsymbol{u}_\epsilon)$ be a sequence in $\boldsymbol{C}^\infty_0(\Omega)$ such that $\boldsymbol{u}_\epsilon(\boldsymbol{x}) \rightharpoonup \boldsymbol{u}(\boldsymbol{x}, y^1)$ and $\boldsymbol{D}\boldsymbol{u}_\epsilon(\boldsymbol{x}) \rightharpoonup \boldsymbol{z}(\boldsymbol{x}, y^1)$. The weak two-scale limit $\boldsymbol{u}$, which belongs to $\boldsymbol{W}^{1,2}_0(\Omega)$, is then independent of y^1, that is, $\boldsymbol{u}(\boldsymbol{x}, y^1) = \boldsymbol{u}(\boldsymbol{x}) \in \boldsymbol{W}^{1,2}_0(\Omega)$. Furthermore, $\boldsymbol{z}(\boldsymbol{x}, y^1) = \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x}, y^1)$, having $\boldsymbol{v} \in \mathbb{L}^2(\Omega, \mathbb{V}^2_{pot})$.*

Proof. Note that our case involves second order tensors. We take $\boldsymbol{h} \in \boldsymbol{L}^2_{\text{per}}(Y^1)$ and $b \in L^2_{\text{per}}(Y^1)$ such that $b = \operatorname{div}\boldsymbol{h}$. The identity (14) means that $\forall \psi \in C^\infty_0(\Omega)$,

$$\epsilon \int_\Omega \boldsymbol{D}\psi(\boldsymbol{x}) \cdot \boldsymbol{h}(\epsilon^{-1}x^1)\,dx = -\int_\Omega \psi(\boldsymbol{x}) b(\epsilon^{-1}x^1)\,dx. \tag{17}$$

Letting $\varphi \in C^\infty_0(\Omega)$, we now use partial differentiation:

$$\int_\Omega \boldsymbol{D}(\varphi(\boldsymbol{x}) u_\epsilon(\boldsymbol{x})) \cdot \boldsymbol{h}(\epsilon^{-1}x^1)\,dx = \int_\Omega (u_\epsilon \boldsymbol{D}\varphi(\boldsymbol{x}) + \varphi(\boldsymbol{x}) \boldsymbol{D}u_\epsilon) \cdot \boldsymbol{h}(\epsilon^{-1}x^1)\,dx. \tag{18}$$

This along with (17) implies that

$$-\int_\Omega (\varphi(\boldsymbol{x}) u_\epsilon(\boldsymbol{x})) b(\epsilon^{-1}x^1)\,dx = \epsilon \int_\Omega u_\epsilon(\boldsymbol{D}\varphi(\boldsymbol{x}) \cdot \boldsymbol{h}(\epsilon^{-1}x^1))\,dx + \epsilon \int_\Omega \boldsymbol{D}u_\epsilon \cdot (\varphi(\boldsymbol{x}) \boldsymbol{h}(\epsilon^{-1}x^1))\,dx.$$

The right hand side goes to zero when $\epsilon \to 0$ because $(\boldsymbol{u}_\epsilon)$ and $(\boldsymbol{D}\boldsymbol{u}_\epsilon)$ two-scale converge weakly (using assumption). As a result, by proceeding to the limit component-wise under the assumption that $\boldsymbol{u}_\epsilon(\boldsymbol{x}) \rightharpoonup \boldsymbol{u}(\boldsymbol{x}, y^1)$, we get

$$\int_{\Omega \times Y^1} u(\boldsymbol{x}, y^1)(\varphi(\boldsymbol{x}) b(y^1))\,dx\,dy^1 = 0.$$

For $\boldsymbol{h} \in \boldsymbol{L}^2(Y^1)$, it follows from Theorem 5.1 that the collection of functions $b \in L^2(Y^1)$ represented by $b = \operatorname{div}\boldsymbol{h}$ (thus $\int_{Y^1} b(y^1)\,dy^1 = 0$) is dense in the subspace of functions in $L^2(Y^1)$ having mean value 0. Thus, u is independent of y^1, that is, $u(\boldsymbol{x}, y^1) = u(\boldsymbol{x})$.

Afterwards, we show that $\boldsymbol{u} \in \boldsymbol{W}_0^{1,2}(\Omega)$ and that $\boldsymbol{z}(\boldsymbol{x}, y^1) = \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x}, y^1)$, in which $\boldsymbol{v} \in \mathbb{L}^2(\Omega, \mathbb{V}^2_{\text{pot}})$. For $\boldsymbol{h} \in \mathbb{V}^2_{\text{sol}}$ with

$$\int_{Y^1} \boldsymbol{h}\, dy^1 = \boldsymbol{\eta}, \tag{19}$$

and $\phi \in C^\infty(\bar{\Omega})$, we obtain the identity

$$\begin{aligned} 0 &= \int_\Omega \boldsymbol{D}(\phi \boldsymbol{u}_\epsilon) \cdot \boldsymbol{h}(\epsilon^{-1} x^1)\, dx \\ &= \int_\Omega (\phi \boldsymbol{D}\boldsymbol{u}_\epsilon + \boldsymbol{u}_\epsilon \otimes \boldsymbol{D}\phi) \cdot (\boldsymbol{h}(\epsilon^{-1} x^1)\, dx \\ &= \int_\Omega \boldsymbol{D}\boldsymbol{u}_\epsilon \cdot (\boldsymbol{h}(\epsilon^{-1} x^1)\phi(\boldsymbol{x}))\, dx \\ &+ \int_\Omega \boldsymbol{u}_\epsilon \cdot (\boldsymbol{h}(\epsilon^{-1} x^1) \boldsymbol{D}\phi(\boldsymbol{x}))\, dx. \end{aligned} \tag{20}$$

Proceeding to the limit in the weak two-scale sense component-wise, we reach

$$\begin{aligned} 0 &= \int_{\Omega \times Y^1} \boldsymbol{u}(\boldsymbol{x}) \cdot (\boldsymbol{h}(y^1) \boldsymbol{D}\phi(\boldsymbol{x}))\, dx\, dy^1 \\ &+ \int_{\Omega \times Y^1} \boldsymbol{z}(\boldsymbol{x}, y^1) \cdot (\boldsymbol{h}(y^1)\phi(\boldsymbol{x}))\, dx\, dy^1. \end{aligned} \tag{21}$$

Using (19), we get $\forall \phi \in C^\infty(\bar{\Omega})$,

$$\int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot (\boldsymbol{\eta} \boldsymbol{D}\phi(\boldsymbol{x}))\, dx = -\int_\Omega v_h(\boldsymbol{x})\phi(\boldsymbol{x})\, dx, \tag{22}$$

where $v_h(\boldsymbol{x}) = \int_{Y^1} \boldsymbol{z}(\boldsymbol{x}, y^1) \cdot \boldsymbol{h}(y^1)\, dy^1$. It follows from (22) that there exists $(h_{ij}) \in \mathbb{V}^2_{\text{sol}}$ such that

$$\int_\Omega \boldsymbol{u}_j(\boldsymbol{x}) \boldsymbol{D}_i \phi(\boldsymbol{x})\, dx = -\int_\Omega v_{h_{ij}}(\boldsymbol{x})\phi(\boldsymbol{x})\, dx, \tag{23}$$

forall $\phi \in C^\infty(\bar{\Omega})$, $\quad i = 1, 2$. Thus, the distributional partial derivatives $\boldsymbol{D}_i \boldsymbol{u}_j = v_{h_{ij}}$ of $\boldsymbol{u}$ are in $L^2(\Omega)$, that is, $\boldsymbol{u} \in \boldsymbol{W}^{1,2}(\Omega)$. Furthermore, equation (23) along with the formula of integration by parts leads to $\boldsymbol{u} \in \boldsymbol{W}_0^{1,2}(\Omega)$ for Ω having Lipschitz property. Now, the equality (21) can be expressed as

$$\begin{aligned} &\int_{Y^1} \int_\Omega \boldsymbol{z}(\boldsymbol{x}, y^1) \cdot (\boldsymbol{h}(y^1)\phi(\boldsymbol{x}))\, dx\, dy^1 \\ &= -\int_{Y^1} \int_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot (\boldsymbol{h}(y^1) \boldsymbol{D}\phi(x))\, dx\, dy^1 \\ &= \int_{Y^1} \int_\Omega \phi(\boldsymbol{x}) \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{h}(y^1)\, dx\, dy^1. \end{aligned}$$

(The right hand side was derived via integrating by parts component-wise.) Hence,

$$\int_{Y^1} \int_\Omega [\boldsymbol{z}(\boldsymbol{x}, y^1) - \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x})] \cdot (\phi(\boldsymbol{x}) \boldsymbol{h}(y^1))\, dx\, dy^1 = 0.$$

Since $\mathbb{L}^2(\Omega, \mathbb{V}^2_{\text{sol}})$ is identified as closure in $\mathbb{L}^2(\Omega \times Y^1)$ of the linear span of matrices $g(\boldsymbol{x})\boldsymbol{h}(y^1)$, where $g \in C_0^\infty(\Omega)$ and $\boldsymbol{h} \in \mathbb{V}^2_{\text{sol}}$, it holds that

$$\int_{Y^1} \int_\Omega [\boldsymbol{z}(\boldsymbol{x}, y^1) - \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x})] \cdot \boldsymbol{w}(\boldsymbol{x}, y^1)\, dx\, dy^1 = 0,$$

for all $\boldsymbol{w} \in \mathbb{L}^2(\Omega, \mathbb{V}^2_{\text{sol}})$. From (11), it is clear that $[\boldsymbol{z}(\boldsymbol{x}, y^1) - \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x})]$ is in $\mathbb{L}^2(\Omega, \mathbb{V}^2_{\text{pot}})$, alternatively, $\boldsymbol{z}(\boldsymbol{x}, y^1) = \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x}, y^1)$, having $\boldsymbol{v} \in \mathbb{L}^2(\Omega, \mathbb{V}^2_{\text{pot}})$.

Sequences $(\boldsymbol{u}_\epsilon)$ in $\boldsymbol{C}_0^\infty(\Omega)$ have been our focus thus far. Nevertheless, everything is true also for sequences $(\boldsymbol{u}_\epsilon)$ in the variable Sobolev space $\boldsymbol{W}_0^{1,2}(\Omega)$, where $\boldsymbol{W}_0^{1,2}(\Omega)$ is determined as the closure of the set of pairs $(\boldsymbol{u}, \boldsymbol{D}\boldsymbol{u})$, where $\boldsymbol{u} \in \boldsymbol{C}_0^\infty(\Omega)$, in $\boldsymbol{L}^2(Y) \times \mathbb{L}^2(Y)$.

Note that this theorem is in two-dimensional elasticity, as a special application. More general n-dimensional results can be found in [5].

References

[1] Gabriel Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM Journal on Mathematical Analysis*, 20(3):608–623, 1989.

[2] Mikhaila Cherdantsev, Kirillb Cherednichenko, and Stefan Neukamm. High contrast homogenisation in nonlinear elasticity under small loads. *Asymptotic Analysis*, 104(1-2):67–102, 2017. DOI: 10.3233/ASY-171430.

[3] Mustapha El Jarroudi. Homogenization of a nonlinear elastic fibre-reinforced composite: A second gradient nonlinear elastic material. *Journal of Mathematical Analysis and Applications*, 403(2):487 – 505, 2013.

[4] V. V. Zhikov and G. A. Yosifian. Introduction to the theory of two-scale convergence. *Journal of Mathematical Sciences*, 197(3):325–357, Mar 2014.

[5] Dag Lukkassen and Peter Wall. Two-scale convergence with respect to measures and homogenization of monotone operators. *Journal of function spaces and applications*, 3(2):125–161, 2005.

[6] Hervé Le Dret. An example of H^1-unboundedness of solutions to strongly elliptic systems of partial differential equations in a laminated geometry. *Proceedings of the Royal Society of Edinburgh*, 105(1):77–82, 1987. doi:10.1017/S0308210500021922.

[7] S. Nekhlaoui, A. Qaiss, M.O. Bensalah, and A. Lekhder. A new technique of laminated composites homogenization. *Adv. Theor. Appl. Mech.*, 3:253–261, 2010.

[8] A. El Omri, A. Fennan, F. Sidoroff, and A. Hihi. Elastic-plastic homogenization for layered composites. *European Journal of Mechanics - A/Solids*, 19(4):585 – 601, 2000.

[9] G. A. Pavliotis and A. M. Stuart. *Multiscale methods: Averaging and homogenization*, volume 53 of *Texts in Applied Mathematics*. Springer-Verlag New York, 2008.

[10] Giuseppe Geymonat, Stefan Müller, and Nicolas Triantafyllidis. Homogenization of nonlinearly elastic materials, microscopic bifurcation and macroscopic loss of rank-one convexity. *Archive for Rational Mechanics and Analysis*, 122(3):231–290, 1993. DOI: 10.1007/BF00380256.

[11] Tina Mai and Jay R. Walton. On strong ellipticity for implicit and strain-limiting theories of elasticity. *Mathematics and Mechanics of Solids*, 20(II):121–139, 2015. DOI: 10.1177/1081286514544254.

[12] Tina Mai and Jay R. Walton. On monotonicity for strain-limiting theories of elasticity. *Journal of Elasticity*, 120(I):39–65, 2015. DOI: 10.1007/s10659-014-9503-4.

[13] Shubin Fu, Eric Chung, and Tina Mai. Generalized multiscale finite element method for a strain-limiting nonlinear elasticity model. *Journal of Computational and Applied Mathematics*, 359:153 – 165, 2019.

[14] Dag Lukkassen, Gabriel Nguetseng, and Peter Wall. Two-scale convergence. *International journal of pure and applied mathematics*, 2(1):35–86, 2002.

[15] Walter Rudin. *Functional analysis, second edition*. International series in pure and applied mathematics. McGraw-Hill, Inc., 1991.

[16] V. V. Zhikov, M. E. Rychago, and S. B. Shul'ga. Homogenization of monotone operators by the method of two-scale convergence. *Journal of Mathematical Sciences*, 127(5):2159–2173, Jun 2005.