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Some results relate to the maximum principle of the subharmonic functions on the unit disc

Một số kết quả về nguyên lý cực đại của hàm điều hòa dưới trên đĩa đơn vị

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Abstract

In this note, we apply the maximum principle of subharmonic functions on the complex plane to prove some results related to the holomorphic functions and the subharmonic functions on unit disc in the complex plane.

Keywords: complex variable functions, holomorphic functions, subharmonic functions, complex analysis.

Tóm tắt

Trong bài báo này, chúng tôi áp dụng nguyên lý cực đại cho hàm điều hòa dưới trên mặt phẳng phức để chứng minh một số kết quả liên quan tới các hàm chỉnh hình và hàm điều hòa dưới xác định trong đĩa đơn vị trên mặt phẳng phức.

Từ khóa: hàm biến phức, hàm chỉnh hình, hàm điều hòa dưới, giải tích phức.

1. Introduction

In potential theory, the subharmonic functions are usually defined on the open set in $\mathbb{R}^n$ (see [1]). This is an advantage for using analytic tools of many variable functions. However, it does not take advantage of the complex number and complex variable function theory. On the other hand, it is hard to extend to the pluripotential theory (see [2], [3]). Theorem 2.2 gives the relation between the holomorphic functions and subharmonic functions. This allows using the complex analytic tools when we study the subharmonic functions on the complex plane.

The maximum principle of subharmonic functions is an interesting topic in potential theory. This principle is established by Phragmén and Lindelöf in [4]. The potential

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theory is a branch of complex analysis that is concentrated to study in the near decades. The maximum principle is established and proved depending on the topology on the extended complex plane (Theorem 2.3). Since the extended complex plane $\mathbb{C}_\infty$ is homeomorphic to the Riemann sphere in the metric space $\mathbb{R}^3$, the extended complex plane $\mathbb{C}_\infty$ is a compact set. This has made the proof of the maximum principle quite simple.

The main aim of this paper is to use the maximum principle of the subharmonic function to prove some results of the holomorphic functions and subharmonic functions on the unit disc in the complex plane (Lemma 3.1, Theorem 3.2 and Theorem 3.4).

2. Preliminaries

We denote by $\mathbb{C}$ the set of all complex numbers (or the complex plane). Let $\mathbb{C}_\infty$ be the extended complex plane that is homeomorphic to the Riemann sphere in the metric space $\mathbb{R}^3$ (see [6]). Since the Riemann sphere is a compact set in $\mathbb{R}^3$, $\mathbb{C}_\infty$ is a compact set.

In this note, we assume the domain to be an open and connected set in $\mathbb{C}$ or $\mathbb{C}_\infty$. Let D be a domain then the closure $\overline{D}$ always takes in $\mathbb{C}_\infty$. Thus, if D is an unbounded domain in $\mathbb{C}$, then $\infty \in D$ and in $\mathbb{C}_\infty$, $\overline{D}$ is a compact set. We also denote $\Delta(\omega,\rho)$ as a disc in $\mathbb{C}$, that is

$$\Delta(\omega,\rho) = \{z \in \mathbb{C}: |z-\omega| < \rho\}.$$

Definition 2.1 (see [1,2,3]) Let U be an open set in $\mathbb{C}$. The function $u: U \to [-\infty,\infty)$ is called subharmonic if it is an upper semicontinuous function and satisfies the local submean inequality, that is, for all $w \in U$ there exists $\rho > 0$ such that

$$u(\omega) \le \frac{1}{2\pi}\int_0^{2\pi} u(\omega + re^{i\theta})d\theta, \qquad (0 \le r < \rho). \qquad (1)$$

The function $v: U \to (-\infty,\infty]$ is superharmonic if the function $-v$ is subharmonic.

We let $SH(U)$ be the set of all subharmonic functions on U. The submean inequality (1) is local, i.e., the number ρ depends on w. Hence, the subharmonicity also has local property, that is, if $(U_\alpha)_{\alpha\in I}$ is an open cover of U, then the function u is a subharmonic function on U if only if it is a subharmonic function on every U_α.

The following result is the relation between the holomorphic function and the subharmonic function.

Theorem 2.2 Let f be a holomorphic function on an open set U in $\mathbb{C}$. Then $\log|f|$ is a subharmonic function on U.

Proof: See Proposition 1.2.23 in [2].

The following result is in [5], we cite it here for the convenience of the reader.

Theorem 2.3 (The maximum principle) Let u be a subharmonic function on the domain D in $\mathbb{C}$. Then we have

a. If u has global extremum on D, then u is constant on D.

b. If $limsup_{z\to\xi} u(x) \le 0$ for all $\xi \in \partial D$, then $u \le 0$ on D.

Proof: **a.** Suppose that u has global extremum value M on D, i.e., there exists $z_0 \in D$ such that

$$u(z) \le M, \forall z \in D \text{ and } u(z_0) = M.$$

Set

$$A = \{z \in D: u(z) < M\}$$

and

$$B = \{z \in D: u(z) = M\}.$$

Then by the semicontinuous of u, we infer that A is open. We prove that B is also open. Indeed, take $\omega \in B$, by Definition 1, there exist $\rho > 0$ such that

$$M = u(\omega) \le \frac{1}{2\pi}\int_0^{2\pi} u(\omega + re^{it})dt \le M$$

for all $0 \le r < \rho$. Infer that

$$\frac{1}{2\pi}\int_0^{2\pi} u(\omega + re^{it})dt = M, \qquad \forall 0 \le r < \rho.$$

Since $u(\omega + re^{it})dt \le M$ for all $r \in [0,\rho)$ and for all $t \in [0,2\pi)$, we have $u(\omega + re^{it})dt = M$, $\forall 0 \le r < \rho$ and $\forall 0 \le t < 2\pi$. So $\Delta(\omega,\rho) \subset B$ and so B is open. Therefore, we have A and B be an open partition of D. Since D is a connected set, we infer either $A = D$ or $B = D$. Because $B \ne \emptyset$ $(z_0 \in B)$ so $B = D$. Thus, we conclude that $u = M$ on D.

b. We extend the function u to the boundary ∂D by setting

$$u(\xi) := \lim_{z\to\xi} \sup u(z) \quad (\xi \in \partial D).$$

Then u is the semicontinuous function on $\bar{D}$. Since $\bar{D}$ is a compact set, u has maximum at some $\omega \in \bar{D}$. If $\omega \in \partial D$, then by assuming we have $u(\omega) \le 0$ and so $u \le 0$ on D. If $\omega \in D$, then by the part a., u is constantly on D and so on $\bar{D}$. This infers that $u \le 0$ on D.

Remark 2.4 In Theorem 3(a), if u has the local extremum or the global minimum on D, then the conclusion is failed. Example: Let $u(z) = max(Rez, 0)$ on $\mathbb{C}$. Then u is the subharmonic function on $\mathbb{C}$. Moreover, u has the local extremum and the global minimum on $\mathbb{C}$, but u is not a constant on $\mathbb{C}$.

3. Main results

In this section, we apply the maximum principle to prove some results for the subharmonic functions and holomorphic functions on the unit disc. These results come from some questions in [5]. First, we have the lemma as follows.

Lemma 3.1 Let u be a subharmonic function on $\Delta(0,1)$ such that $u < 0$. Then for all $\xi \in \partial\Delta(0,1)$ we have

$$\lim_{r\to 1^-} \frac{u(r\xi)}{1-r} < 0.$$

Proof: Set $v(z) = u(z) + c\log|z|$ (here c is a positive constant) on $A = \left\{\frac{1}{2} < |z| < 1\right\}$. Then we have

• The function v is a subharmonic function on A (by Theorem 2.2).

• For all $|\xi| = 1$, we have $\lim_{z\to\xi} v(z) \le 0$.

To apply the maximum principle (Theorem 2.3) to the function v on A, we need to find c such that for all $|x| = \frac{1}{2}$, we have

$$\lim_{z\to\xi} \sup u(z) \le 0.$$

Set $\lambda = \sup\left\{u(\xi): |\xi| = \frac{1}{2}\right\}$. We infer that $\lambda < 0$.

We have

$$\lim_{z\in A, z\to\xi} v(z) \le \lambda + c\log\frac{1}{2} \le 0.$$

From this inequality, we have $c \ge \frac{\lambda}{log2}$.

Now, with $c \ge \frac{\lambda}{log2}$, applying Theorem 2.3 to the function v we infer

$$v(z) \le 0 \Leftrightarrow u(z) \le -clog|z|, \quad \forall \frac{1}{2} < |z| < 1.$$

Then for all $|\xi| = 1$, we have

$$\lim_{r\to 1^-} \frac{u(r\xi)}{1-r} \le \lim_{r\to 1^-} (-c)\frac{logr}{1-r} = c.$$

From the estimations above, if we choose the constant c such that $\frac{\lambda}{log2} \le c < 0$, then we have the conclusion in the following theorem.

Theorem 3.2 *Set* $\Delta = \Delta(0,1)$. Let $f: \Delta \to \Delta$ *be a holomorphic function such that*

$$f(z) = z + o(|1-z|^3) \quad when\ z \to 1.$$

a. *Let* $\phi(z) = \frac{1+z}{1-z}$
and $u(z) = Re\left(\phi(z) - \phi(f(z))\right)$.

Prove that

$\limsup_{z\to\xi} u(z) \le 0 \quad \forall \xi \in \partial\Delta \setminus \{1\}$,
and $u(z) = o(|1-z|)$ when $z \to 1$.

b. *Prove that* $u \le 0$ *on* Δ.

c. *Prove that* $u \equiv 0$ *on* Δ.

d. *Prove that* $f(z) \equiv z$ *on* Δ.

Proof: a. We have

$$Re\phi(z) = \frac{1}{2}\left(\frac{1+z}{1-z} + \frac{1+\bar{z}}{1-\bar{z}}\right) = \frac{1-|z|^2}{|1-z|^2}.$$

This infers that for every $\xi \in \partial\Delta \setminus \{1\}$ we have

• $limsup_{z\to\xi} Re\phi(z) = 0$.

• For all $z \in \Delta$ then $Re\phi(z) > 0$. So we infer $Re\phi(f(z)) > 0$.

Now, for all $\xi \in \partial\Delta \setminus \{1\}$ we have

$$\lim_{z\to\xi} sup\ u(z) \le \lim_{z\to\xi} \sup \phi(z) = 0.$$

In case $z \to 1$ we have

$$\phi(z) - \phi(f(z)) = \frac{1+z}{1-z} - \frac{1+z+o(|1-z|^3)}{1-z-o(|1-z|^3)}$$

$$= \frac{-(1+z)o(|1-z|^3) - (1-z)o(|1-z|^3)}{(1-z)(1-z-o(|1-z|^3))}$$

$$= \frac{-2.o(|1-z|^3)}{(1-z)(1-z-o(|1-z|^3))} = o(|1-z|).$$

From this we infer

$$u(z) = Re\left(\phi(z) - \phi(f(z))\right) = o(|1-z|)$$
when $z \to 1$.

b. From the above formula, we infer that *u* is a subharmonic function on Δ. By (a), we infer that

$$\lim_{z\to\xi} \sup u(z) \le 0 \qquad \text{for all } \xi \in \partial\Delta.$$

By the maximum principle (Theorem 2.3), we derive *u* ≤ 0 on Δ.

c. By (b), we have $u \le 0$ on Δ.

If $u < 0$ on Δ then by Lemma 3.1, for all $\xi \in \partial\Delta$ we have

$$\lim_{r\to 1^-} sup \frac{u(r\xi)}{1-r} < 0. \qquad (*)$$

When $\xi = 1$, by (a), we have

$$u(r) = o(|1-r|) \qquad \text{when } r \to 1^-.$$

This infers that

$$\lim_{r\to 1^-} sup \frac{u(r)}{1-r} = \lim_{r\to 1^-} sup \frac{o(|1-r|)}{1-r} = 0.$$

This is a contradiction to (*).

So $u \equiv 0$ on Δ.

d. By (c), we have

$$Re\frac{1+z}{1-z} = Re\frac{1+f(z)}{1-f(z)} \qquad \text{on } \Delta.$$

This derives that the function $g(z) := \frac{1+z}{1-z} - \frac{1+f(z)}{1-f(z)}$ is holomorphic on Δ that has real part equal to zero. By the Cauchy-Riemann condition (Theorem 2 in [6]), the imaginary part of *g*(*z*) is constant. So we have $g(z) = ai$. Here, *a* is complex number.

On the other hand, by (a), we have

$$g(z) = o(|1-z|) \qquad \text{when } z \to 1.$$

This infers that $\lim_{z\to 1} g(z) = 0 \ or \ ai = 0$. So we have $a = 0$, i.e $g \equiv 0$ on Δ.

So for all $z \in \Delta$ we have

$$\frac{1+z}{1-z}=\frac{1+f(z)}{1-f(z)} \Leftrightarrow \frac{2}{1-z}=\frac{2}{1-f(z)} \Leftrightarrow f(z)=z.$$

Remark 3.3 In Theorem 3.2, if we suppose that

$$f(z)=z+O(|1-z|^3) \quad \text{when } z\to 1$$

then the conclusion in (d) is failed.

Indeed, considering $f(z)=z+\lambda(1-z)^3$, here $\lambda>0$ small enough. Then with $|z|=1$ we have

$$|f(z)|^2=(z+\lambda(1-z)^3)(\bar{z}+\lambda\overline{(1-z)^3})$$
$$=1-2\lambda Re(\bar{z}.(1-z)^3)+\lambda^2(1-z)^3\overline{(1-z)^3}$$
$$=1+2\lambda[4Rez-3-Rez^2]+8\lambda^2(1-Rez)^3$$

Set $z=cost+isint$ here $0\le t\le 2\pi$. Then to prove that $|f(z)|^2\le 1$ we need the following

$$2\lambda[4Rez-3-Rez^2]+8\lambda^2(1-Rez)^3\le 0 \quad \forall |z|=1.$$

This is equivalent to

$$4\,cost-3-cos2t+4\lambda(1-cost)^3\le 0 \qquad \forall 0\le t\le 2\pi$$
$$\Leftrightarrow (1-cost)2\big(-2+4\lambda(1-cost)\big)\le 0 \qquad \forall 0\le t\le 2\pi.$$

This is true if we choose $0<\lambda<\frac{1}{4}$.

So the function $f:\Delta\to\Delta$ is holomorphic and satisfies $f(z)=z+O(|1-z|^3)$. But *f* is not an identical function.

Theorem 3.4 *Let u be a subharmonic function on* $\Delta(0,1)$ *such that*

$$u(z)\le -\log|Imz| \qquad (|z|<1)$$

Then prove that

$$u(z)\le -\log\left|\frac{1-z^2}{2}\right| \qquad (|z|<1).$$

Proof: With $0<r<1$ we consider the function following

$$v(z)=u(z)+\log\left|\frac{r^2-z^2}{2r}\right|, \qquad z\in\Delta(0,r).$$

Then by Theorem 2.2, *v* is a subharmonic function on $\Delta(0,r)$. Take $\xi\in\partial\Delta(0,r)$. We consider two cases as follows.

• If $\xi\ne r$ and $\xi=r(cost+i\,sint)$ then we have

$$\limsup_{z\to\xi} v(z)=\lim_{z\to\xi}\left(u(z)+\log\left|\frac{r^2-z^2}{2r}\right|\right)$$
$$\le \limsup_{z\to\xi}\log\left|\frac{r^2-z^2}{2r|Imz|}\right|$$
$$=\limsup_{z\to\xi}\log\frac{|r^2-\xi^2|}{2r|Im\xi|}=0.$$

If $\xi = r$ then we have

$$\limsup_{z\to r} v(z) = \limsup_{z\to r} \left(u(z) + \log\left|\frac{r^2 - z^2}{2r}\right|\right) = -\infty.$$

(This is because u is bounded on $\overline{\Delta}(0,r)$. By applying the maximum principle (Theorem 2.3) to function *v* on $\Delta(0,r)$ we infer

$$u(z) \leq -\log\left|\frac{r^2 - z^2}{2r}\right|, \qquad \forall z \in \Delta(0,r).$$

Let $r \to 1-$ we get

$$u(z) \leq -\log\left|\frac{1 - z^2}{2}\right| \qquad \forall z \in \Delta(0,1)$$

4. Conclusion

In this note, we apply the maximum principle of the subharmonic functions to prove some results relate to the boundedness of the holomorphic function and subharmonic functions on the unit disc in the complex plane (Lemma 3.1, Theorem 3.2 and Theorem 3.4).

References

[1] David H. Armitage, Stephen J. Gardiner. (2001). *Classical Potential Theory*. Place of publication: Springer.

[2] P. H. Hiep. (2016). *Singularities of plurisubharmonic functions*. Place of publication: Pub. Hou. Sci. and Tec..

[3] Klimek M. (1991). *Pluripotential Theory*. Place of publication: Clarendon Press, Oxford.

[4] Phragmén E., Lindelöf E. (1908). Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des fonctions monogènes dans le voisinage d'un point singulier. *Acta Math, 31(1)*, 381 – 406. DOI: 10.1007/BF02415450.

[5] Ransford T. (1995). *Potential Theory in Complex plane*. Place of publication: Cambridge University Press.

[6] N. V. Khue, L. M. Hai. (1997). *Ham bien phuc*. Place of publication: ĐHQG Ha Noi.