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TRƯỜNG ĐẠI HỌC DUY TÂN
DUYTAN UNIVERSITY

Numerical homogenization technique for a strain-limiting nonlinear elasticity model

Kỹ thuật đồng nhất hóa số cho một mô hình đàn hồi phi tuyến tính giới hạn biến dạng

Tina Mai[a,b*]
Mai Ti Na[a,b*]

[a] *Institute of Research and Development, Duy Tan University, Da Nang, 550000, Vietnam*
[a] *Viện Nghiên cứu và Phát triển Công nghệ Cao, Trường Đại học DuyTân, Đà Nẵng, Việt Nam*
[b] *Faculty of Natural Sciences, Duy Tan University, Da Nang, 550000, Vietnam*
[b] *Khoa Khoa học Tự nhiên, Trường Đại học Duy Tân, Đà Nẵng, Việt Nam*

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Abstract

We describe a numerical homogenization technique for a two-dimensional nonlinear equation emerging from strain-limiting elasticity.

Keywords: numerical homogenization; two-dimensional; nonlinear equation; strain-limiting elasticity.

Tóm tắt

Chúng tôi trình bày một kỹ thuật đồng nhất hóa số cho một phương trình phi tuyến tính hai chiều phát sinh từ độ đàn hồi giới hạn biến dạng.

Từ khóa: đồng nhất hóa số; hai chiều; phương trình phi tuyến tính; độ đàn hồi giới hạn biến dạng.

1. Introduction

We theoretically describe a popular numerical homogenization method for a two-dimensional nonlinear equation arising from strain-limiting elasticity. According to this strategy, the number of degrees of freedom for each coarse element is limited. Based on [1], our goal is to demonstrate that this numerical homogenization is a finite element approximation on a coarse grid utilizing harmonic extension, where each edge has only one degree of freedom.

2. Preliminaries

Latin indices belong to the set $\{1,2\}$. Italic capitals (e.g. $L^2(\Omega)$), boldface Roman capitals (e.g. $\boldsymbol{V}$), and special Roman capitals (e.g. $\mathbb{S}$) respectively denote the spaces of functions, vector fields in $\mathbb{R}^2$, and 2×2 matrix fields over Ω.

The Sobolev norm $\|\cdot\|_{\boldsymbol{W}_0^{1,2}(\Omega)}$ takes the form

$$\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} = (\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla \boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)})^{\frac{1}{2}}\,;$$

here, $\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} := \||\boldsymbol{v}|\|_{\boldsymbol{L}^2(\Omega)}$, where $|\boldsymbol{v}|$ stands for

Corresponding Author: Tina Mai; *Institute of Research and Development, Duy Tan University, Da Nang, 550000, Vietnam; Faculty of Natural Sciences, Duy Tan University, Da Nang, 550000, Vietnam*
Email: maitina@duytan.edu.vn

the Euclidean norm of the 2-component vector-valued function $\boldsymbol{v}$, and $\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2(\Omega)} := \||\nabla \boldsymbol{v}|\|_{\mathbb{L}^2(\Omega)}$, in which $|\nabla \boldsymbol{v}|$ represents the Frobenius norm of the 2×2 matrix $\nabla \boldsymbol{v}$. Recall that the Frobenius norm on $\mathbb{L}^2(\Omega)$ is expressed by $|\boldsymbol{X}|^2 := \boldsymbol{X} \cdot \boldsymbol{X} = \operatorname{tr}(\boldsymbol{X}^{\mathrm{T}} \boldsymbol{X})$.

2.1. Notations

To go through the key idea, we consider our case from strain-limiting elasticity [2, 3, 4]:

$$-\operatorname{div}(\boldsymbol{\kappa}(x^1, |\boldsymbol{D}\boldsymbol{u}|)\boldsymbol{D}\boldsymbol{u}) = \boldsymbol{f} \text{ in } \Omega, \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega. \tag{1}$$

Equivalently,

$$-\operatorname{div}(\boldsymbol{a}(x^1, \boldsymbol{D}\boldsymbol{u})) = \boldsymbol{f} \text{ in } \Omega, \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega, \tag{2}$$

where $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$,

$$\boldsymbol{a}(x^1, \boldsymbol{D}\boldsymbol{u}) = \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\boldsymbol{u}|)\boldsymbol{D}\boldsymbol{u} = \frac{\boldsymbol{D}\boldsymbol{u}}{1 - \beta(x^1)|\boldsymbol{D}\boldsymbol{u}|}$$

is a high-contrast coefficient, $\boldsymbol{f} \in \boldsymbol{H}_*^1(\Omega) \subset \boldsymbol{L}^2(\Omega) \subsetneq \boldsymbol{H}^{-1}(\Omega)$ is an external forcing term, $\boldsymbol{a}(x^1, \cdot)$ is assumed to be very heterogeneous with respect to $\boldsymbol{x} = (x^1, x^2)$.

Let

$$\mathcal{Z} := \left\{ \boldsymbol{\zeta} \in \mathbb{L}^2(\Omega) \mid 0 \leq |\boldsymbol{\zeta}| < \frac{1}{\beta(x^1)} < 1 \right\}, \tag{3}$$

and let

$$\mathcal{U} = \{\boldsymbol{w} \in \boldsymbol{H}^1(\Omega) \mid \boldsymbol{D}\boldsymbol{w} \in \mathcal{Z}\}, \tag{4}$$

with the given $\mathcal{Z}$ in (3).

Remark 2.1. *We shall employ the condition $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ or $\boldsymbol{H}^1(\Omega)$ (depending on the context) meaning that $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{U}$ for the remainder of the paper, without misunderstanding.*

Following is the corresponding weak formulation: $(\mathscr{P})$ Find $\boldsymbol{u}$ in $\boldsymbol{H}_0^1(\Omega)$ such that

$$\int_\Omega \boldsymbol{a}(x^1, \boldsymbol{D}\boldsymbol{u}) \cdot \boldsymbol{D}\boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}, \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{5}$$

It has been proved in [5, 6] that $(\mathscr{P})$ with (5) is well-posed. The energy norm of $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$ is referred to as

$$\|\boldsymbol{u}\|_{1,2(\Omega)} = \left(\int_\Omega \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\boldsymbol{u}|)|\boldsymbol{D}\boldsymbol{u}|^2 \, dx \right)^{1/2}. \tag{6}$$

Next, we describe the approximate solution using finite elements [1]. Suppose $\mathcal{T}^h$ is a fine triangulation. With regard to $\mathcal{T}^h$, let $\boldsymbol{V}^h = \boldsymbol{V}^h(\Omega)$ be the standard finite element space that contains continuous piecewise linear functions. The subset of $\boldsymbol{V}^h(\Omega)$ consisting of functions that vanish on $\partial\Omega$ is also denoted by $\boldsymbol{V}_0^h(\Omega) = \boldsymbol{V}_0^h$. Following is a definition of the discrete fine-scale problem: $(\mathscr{P}^h)$ Find $\boldsymbol{u}^h \in \boldsymbol{V}^h(\Omega)$ such that

$$\int_\Omega \boldsymbol{a}(x^1, \boldsymbol{D}\boldsymbol{u}^h) \cdot \boldsymbol{D}\boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_0^h(\Omega). \tag{7}$$

We also present a coarse discretization $\mathcal{T}^H$, where each coarse block is made up of a localized fine mesh. Figure 1 in [1] provides an example of a multiscale discretization with both fine and coarse elements. Now, let us denote the vertices of the coarse grid by $\{\boldsymbol{x}_i\}_{i=1}^{N_v}$ and construct a coarse neighborhood $\boldsymbol{x}_i$ by

$$w_i = \cup\{K_j \in \mathcal{T}^H;\ x_i \in \bar{K}_j\}, \tag{8}$$

in which N_v is the number of coarse vertices, and K_j represents the coarse block in the domain. Within each coarse neighborhood w_i $(i = 1, \cdots, N_v)$, the set of coarse edges with a common vertex $\boldsymbol{x}_i$ is called the *cross* of $\boldsymbol{x}_i$.

3. Harmonic extension

We first introduce the notion of 2-*harmonic extension* [1], or $\boldsymbol{a}$-*harmonic extension* ([7]), or for short, *harmonic extension*, or *extension*.

Definition 3.1. *Provided K and $\boldsymbol{u} \in \boldsymbol{H}^1(K)$, let $\tilde{\boldsymbol{u}} \in \boldsymbol{H}^1(K)$ be defined so that $\tilde{\boldsymbol{u}} - \boldsymbol{u} \in \boldsymbol{H}_0^1(K)$ and that $\tilde{\boldsymbol{u}}$ satisfies*

$$-\operatorname{div}(\boldsymbol{a}(x^1, \boldsymbol{D}\tilde{\boldsymbol{u}})) = \boldsymbol{0} \text{ in } K, \tag{9}$$

*in which $\boldsymbol{a}(x^1, \boldsymbol{D}\tilde{\boldsymbol{u}}) = \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\tilde{\boldsymbol{u}}|)\boldsymbol{D}\tilde{\boldsymbol{u}}$. Then, $\tilde{\boldsymbol{u}}$ is called the **2-harmonic extension** or **a-harmonic extension** of $\boldsymbol{u}$ and denoted by $\boldsymbol{H}_2(\boldsymbol{u})$.*

Notice that the weak form of (9) is

$$\int_\Omega \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\tilde{\boldsymbol{u}}|)\boldsymbol{D}\tilde{\boldsymbol{u}} \cdot \boldsymbol{D}\boldsymbol{v} \, dx = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{10}$$

Taking $\boldsymbol{v} = \tilde{\boldsymbol{u}}$ in (10), we get

$$\int_\Omega \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\tilde{\boldsymbol{u}}|)|\boldsymbol{D}\tilde{\boldsymbol{u}}|^2 \, dx = 0. \tag{11}$$

Remark 3.2. *The harmonic extension minimizes the energy norm, that is,*

$$\int_K \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\tilde{\boldsymbol{u}}|)|\boldsymbol{D}\tilde{\boldsymbol{u}}|^2\,dx \tag{12}$$

$$= \min_{\boldsymbol{v}\in \boldsymbol{H}^1_{\boldsymbol{u}}(K)} \int_K \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\boldsymbol{v}|)|\boldsymbol{D}\boldsymbol{v}|^2\,dx, \tag{13}$$

with $\boldsymbol{H}^1_{\boldsymbol{u}}(K) = \{\boldsymbol{v} \in \boldsymbol{H}^1(K) \mid \boldsymbol{v} = \boldsymbol{u}$ *on* $\partial K\}$.

Remark 3.3. *In our paper, all harmonic extensions are attained coarse-element by coarse-element* K*. Even though we might use the notation* $\boldsymbol{H}_2$ *directly on a bigger domain such as a coarse neighborhood* w_i *or the whole domain* Ω*, it implies that the extension is operated on each coarse element* K *belonging to* w_i *or* Ω*.*

4. Numerical Homogenization (NH)

Based on [1], we consider

$$-\mathrm{div}(\boldsymbol{a}(x^1, \boldsymbol{D}\boldsymbol{u})) = \boldsymbol{f} \text{ in } \Omega,$$

having $\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega)$ (that is $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$). For each coarse-grid block K, our goal is to compute the effective property. This is accomplished by solving the local problem

$$-\mathrm{div}(\boldsymbol{a}(x^1, \boldsymbol{D}\boldsymbol{N}_{\boldsymbol{\xi}})) = \boldsymbol{0} \text{ in } K,$$

where the boundary condition is $\boldsymbol{N}_{\boldsymbol{\xi}} = \boldsymbol{\xi}\boldsymbol{x}$ on ∂K. By the Definition of harmonic extension 3.1, we can express $\boldsymbol{N}_{\boldsymbol{\xi}} = \boldsymbol{H}_2(\boldsymbol{\xi}\boldsymbol{x})$. Thus, $\boldsymbol{a}_*(\cdot)$ is defined as follows:

$$\boldsymbol{a}_*(\boldsymbol{\xi}) = \frac{1}{|K|}\int_K \boldsymbol{a}(y^1, \boldsymbol{D}\boldsymbol{N}_{\boldsymbol{\xi}})\,dy.$$

The coarse-grid equation is then of the form

$$-\mathrm{div}(\boldsymbol{a}_*(\boldsymbol{D}\boldsymbol{u}_*)) = \boldsymbol{f} \text{ in } \Omega,$$

where $\boldsymbol{u}_* \in \boldsymbol{H}^1_0(\Omega)$. We assume that $\boldsymbol{u}_* = \sum c_k \boldsymbol{\phi}_k$ (in which $\{\boldsymbol{\phi}_k\}$ is a linear basis) to get

$$F^{NH}(\vec{\boldsymbol{c}}) = \int_\Omega \boldsymbol{a}_*(\boldsymbol{D}\sum c_k\boldsymbol{\phi}_k)\cdot \boldsymbol{D}\boldsymbol{\phi}_j\,dx$$
$$= \sum_{K\in\Omega}\int_K \boldsymbol{a}_*(\sum c_k \boldsymbol{D}\boldsymbol{\phi}_k)\cdot \boldsymbol{D}\boldsymbol{\phi}_j\,dx.$$

At this stage, denoting $\sum c_k \boldsymbol{D}\boldsymbol{\phi}_k = \boldsymbol{\xi} = \boldsymbol{constant}$, we obtain $\boldsymbol{N}_{\boldsymbol{\xi}} = \boldsymbol{H}_2(\sum c_k(\boldsymbol{D}\boldsymbol{\phi}_k)\boldsymbol{x})$ and

$$\begin{aligned}
&F^{NH}(\vec{\boldsymbol{c}})\\
&= \sum_{K\in\Omega}\int_K \boldsymbol{a}_*(\boldsymbol{\xi})\cdot \boldsymbol{D}\boldsymbol{\phi}_j\,dx\\
&= \sum_{K\in\Omega}\int_K \left(\frac{1}{|K|}\int_K \boldsymbol{a}(x^1, \boldsymbol{D}\boldsymbol{N}_{\boldsymbol{\xi}})\,dx\right)\cdot \boldsymbol{D}\boldsymbol{\phi}_j\,dx\\
&= \int_\Omega \frac{1}{|K|}\left(\int_K \boldsymbol{a}(x^1, \boldsymbol{D}\boldsymbol{H}_2(\sum c_k(\boldsymbol{D}\boldsymbol{\phi}_k)\boldsymbol{x}))\,dx\right)\cdot \boldsymbol{D}\boldsymbol{\phi}_j\,dx\\
&= \int_\Omega \frac{1}{|K|}\left(\int_K \boldsymbol{a}(x^1, \boldsymbol{D}\boldsymbol{H}_2(\sum c_k\boldsymbol{\phi}_k))\,dx\right)\cdot \boldsymbol{D}\boldsymbol{\phi}_j\,dx\\
&= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\phi}_j\,dx.
\end{aligned}$$

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