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Multiscale model reduction approach for a nonlinear equation arising from elasticity

Phương pháp giảm thiểu mô hình đa kích thước cho một phương trình phi tuyến tính phát sinh từ độ đàn hồi

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Abstract

We present a multiscale model reduction framework utilizing generalized multiscale finite element method, for a two-dimensional nonlinear equation emerging from strain-limiting elasticity.

Keywords: generalized multiscale finite element method; two-dimensional; nonlinear equation; strain-limiting elasticity

Tóm tắt

Chúng tôi trình bày một khuôn khổ giảm thiểu mô hình đa kích thước dùng phương pháp phần tử hữu hạn đa kích thước tổng quát, cho một phương trình phi tuyến tính hai chiều phát sinh từ độ đàn hồi giới hạn biến dạng.

Từ khóa: phương pháp phần tử hữu hạn đa kích thước tổng quát; hai chiều; phương trình phi tuyến tính; độ đàn hồi giới hạn biến dạng

1. Introduction

Based on [1], we theoretically present a multiscale model reduction framework using generalized multiscale finite element method (GMsFEM) for a nonlinear equation occurring in strain-limiting elasticity. In particular, we seek the generalized multiscale finite element solution of such equation on the coarse grid's *crosses* utilizing nonlinear harmonic functions [1]. Note that the ability to capture the impact of small and separable scales makes studying nonlinear functions crucial. By extending the acquired cross values into the entire domain, the global solution may then be approximated. The Numerical Homogenization (NH) [1] is what inspired this concept. Our goal is to demonstrate that the suggested generalized multiscale finite element method recovers NH.

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2. Preliminaries

Latin indices are in the set $\{1,2\}$. Italic capitals (e.g. $L^2(\Omega)$), boldface Roman capitals (e.g. $\boldsymbol{V}$), and special Roman capitals (e.g. $\mathbb{S}$) stand for the spaces of functions, vector fields in $\mathbb{R}^2$, and 2×2 matrix fields over Ω, respectively.

The Sobolev norm $\|\cdot\|_{\boldsymbol{W}_0^{1,2}(\Omega)}$ is expressed by

$$\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,2}(\Omega)} = (\|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\boldsymbol{v}\|^2_{\mathbb{L}^2(\Omega)})^{\frac{1}{2}};$$

in which, $\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} := \||\boldsymbol{v}|\|_{L^2(\Omega)}$, where $|\boldsymbol{v}|$ symbolizes the Euclidean norm of the 2-component vector-valued function $\boldsymbol{v}$, and $\|\nabla\boldsymbol{v}\|_{\mathbb{L}^2(\Omega)} := \||\nabla\boldsymbol{v}|\|_{\mathbb{L}^2(\Omega)}$, where $|\nabla\boldsymbol{v}|$ designates the Frobenius norm of the 2×2 matrix $\nabla\boldsymbol{v}$. Recall that the Frobenius norm on $\mathbb{L}^2(\Omega)$ has the form $|\boldsymbol{X}|^2 := \boldsymbol{X}\cdot\boldsymbol{X} = \mathrm{tr}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X})$.

2.1. Notations

To explain the main concept, we consider the following case from strain-limiting elasticity [2, 3, 4]:

$$-\mathrm{div}(\boldsymbol{\kappa}(x^1,|\boldsymbol{D}\boldsymbol{u}|)\boldsymbol{D}\boldsymbol{u}) = \boldsymbol{f} \text{ in } \Omega, \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega. \tag{1}$$

Equivalently,

$$-\mathrm{div}(\boldsymbol{a}(x^1,\boldsymbol{D}\boldsymbol{u})) = \boldsymbol{f} \text{ in } \Omega, \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega, \tag{2}$$

in which $\boldsymbol{u}\in\boldsymbol{H}_0^1(\Omega)$,

$$\boldsymbol{a}(x^1,\boldsymbol{D}\boldsymbol{u}) = \boldsymbol{\kappa}(x^1,|\boldsymbol{D}\boldsymbol{u}|)\boldsymbol{D}\boldsymbol{u} = \frac{\boldsymbol{D}\boldsymbol{u}}{1-\beta(x^1)|\boldsymbol{D}\boldsymbol{u}|}$$

is a high-contrast coefficient $\boldsymbol{a}(x^1,\cdot)$ and assumed to be very heterogeneous with respect to $\boldsymbol{x} = (x^1,x^2)$, and $\boldsymbol{f}\in\boldsymbol{H}^1_*(\Omega)\subset\boldsymbol{L}^2(\Omega)\subsetneq\boldsymbol{H}^{-1}(\Omega)$ is an external forcing term.

Let

$$\mathcal{Z} := \left\{\boldsymbol{\zeta}\in\mathbb{L}^2(\Omega)\mid 0\le|\boldsymbol{\zeta}|<\frac{1}{\beta(x^1)}<1\right\}, \tag{3}$$

and let

$$\mathscr{U} = \{\boldsymbol{w}\in\boldsymbol{H}^1(\Omega)\mid \boldsymbol{D}\boldsymbol{w}\in\mathcal{Z}\}, \tag{4}$$

with the provided $\mathcal{Z}$ in (3).

Remark 2.1. *Without ambiguity, we will use the hypothesis $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega)$ or $\boldsymbol{H}^1(\Omega)$ (depending on the context) with the meaning that $\boldsymbol{u},\boldsymbol{v}\in\mathscr{U}$, for the remaining of this paper.*

Here is the equivalent weak formulation: $(\mathscr{P})$ Find $\boldsymbol{u}$ in $\boldsymbol{H}_0^1(\Omega)$ such that

$$\int_\Omega \boldsymbol{a}(x^1,\boldsymbol{D}\boldsymbol{u})\cdot\boldsymbol{D}\boldsymbol{v} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}, \quad \forall\boldsymbol{v}\in\boldsymbol{H}_0^1(\Omega). \tag{5}$$

It has been demonstrated that $(\mathscr{P})$ with (5) is a well-posed problem in [5, 6]. The energy norm of $\boldsymbol{u}\in\boldsymbol{H}^1(\Omega)$ is denoted by

$$\|\boldsymbol{u}\|_{1,2(\Omega)} = \left(\int_\Omega \boldsymbol{\kappa}(x^1,|\boldsymbol{D}\boldsymbol{u}|)|\boldsymbol{D}\boldsymbol{u}|^2\,dx\right)^{1/2}. \tag{6}$$

The approximate solution is then described using finite elements [1]. Let $\mathcal{T}^h$ be a fine triangulation. Also, let $\boldsymbol{V}^h = \boldsymbol{V}^h(\Omega)$ be the classical finite element space (for $\mathcal{T}^h$), which contains continuous piecewise linear functions. The expression $\boldsymbol{V}_0^h(\Omega) = \boldsymbol{V}_0^h$ designates the subset of $\boldsymbol{V}^h(\Omega)$ possessing functions that become zero on $\partial\Omega$. Discrete fine-scale problem definition is provided below: $(\mathscr{P}^h)$ Find $\boldsymbol{u}^h\in\boldsymbol{V}^h(\Omega)$ such that

$$\int_\Omega \boldsymbol{a}(x^1,\boldsymbol{D}\boldsymbol{u}^h)\cdot\boldsymbol{D}\boldsymbol{v} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}, \quad \forall\boldsymbol{v}\in\boldsymbol{V}_0^h(\Omega). \tag{7}$$

Moreover, a coarse discretization is introduced under the name $\mathcal{T}^H$, in which every coarse element is composed of a localized fine grid. An illustration of a multiscale discretization with both fine and coarse elements can be found in [1]'s Figure 1. Now, let us use the notation $\{\boldsymbol{x}_i\}_{i=1}^{N_v}$ to represent the coarse grid's vertices and create a coarse neighborhood $\boldsymbol{x}_i$ by

$$w_i = \cup\{K_j\in\mathcal{T}^H;\ x_i\in\bar{K}_j\}, \tag{8}$$

where K_j denotes a coarse block in the domain Ω and N_v stands for the number of coarse vertices. The collection of coarse edges having a common vertex $\boldsymbol{x}_i$ within each coarse neighborhood w_i $(i=1,\cdots,N_v)$ is referred to as the *cross* of $\boldsymbol{x}_i$.

3. Harmonic extension

We begin by defining the term 2-*harmonic extension* [1], or $\boldsymbol{a}$-*harmonic extension* [7], or *harmonic extension*, often known as *extension*.

Definition 3.1. *Given K and $\boldsymbol{u} \in \boldsymbol{H}^1(K)$, let $\tilde{\boldsymbol{u}} \in \boldsymbol{H}^1(K)$ be defined so that $\tilde{\boldsymbol{u}} - \boldsymbol{u} \in \boldsymbol{H}_0^1(K)$ and $\tilde{\boldsymbol{u}}$ makes the following hold*

$$-\mathrm{div}(\boldsymbol{a}(x^1, \boldsymbol{D}\tilde{\boldsymbol{u}})) = \boldsymbol{0} \text{ in } K, \tag{9}$$

in which $\boldsymbol{a}(x^1, \boldsymbol{D}\tilde{\boldsymbol{u}}) = \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\tilde{\boldsymbol{u}}|)\boldsymbol{D}\tilde{\boldsymbol{u}}$. Then, $\tilde{\boldsymbol{u}}$ is referred to as the ***2-harmonic extension*** *or* ***a-harmonic extension*** *of $\boldsymbol{u}$ and represented by the symbol $\boldsymbol{H}_2(\boldsymbol{u})$.*

Note that (9) has a weak form which is

$$\int_\Omega \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\tilde{\boldsymbol{u}}|)\boldsymbol{D}\tilde{\boldsymbol{u}} \cdot \boldsymbol{D}\boldsymbol{v}\, dx = 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{10}$$

Choosing $\boldsymbol{v} = \tilde{\boldsymbol{u}}$ in (10), there holds

$$\int_\Omega \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\tilde{\boldsymbol{u}}|)|\boldsymbol{D}\tilde{\boldsymbol{u}}|^2\, dx = 0. \tag{11}$$

Remark 3.2. *The harmonic extension minimizes the energy norm, that is*

$$\int_K \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\tilde{\boldsymbol{u}}|)|\boldsymbol{D}\tilde{\boldsymbol{u}}|^2\, dx \tag{12}$$

$$= \min_{\boldsymbol{v} \in \boldsymbol{H}_{\boldsymbol{u}}^1(K)} \int_K \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\boldsymbol{v}|)|\boldsymbol{D}\boldsymbol{v}|^2\, dx, \tag{13}$$

with $\boldsymbol{H}_{\boldsymbol{u}}^1(K) = \{\boldsymbol{v} \in \boldsymbol{H}^1(K) \mid \boldsymbol{v} = \boldsymbol{u} \text{ on } \partial K\}$.

Remark 3.3. *All harmonic extensions are achieved coarse-element by coarse-element K in our paper. The extension is carried out on each coarse element K belonging to w_i or Ω, even though we could use the notation $\boldsymbol{H}_2$ directly on a larger domain like a coarse neighborhood w_i or the entire domain Ω.*

This idea of a nonlinear harmonic function will be applied in the section below.

4. Generalized multiscale finite element method

Based on [1], this approach's name indicates that we want to seek a numerical approximation of the solution and use the degrees of freedom alone on the crosses to demonstrate model reduction. We assume that the generalized multiscale finite element solution needs to be found is of the form

$$\boldsymbol{u}_{ms} = \boldsymbol{H}_2\Big(\sum_i \sum_{k=1}^{L_i} c_k^{w_i} \chi_i \boldsymbol{\phi}_k^{w_i}\Big), \tag{14}$$

where the collection of multiscale basis functions established in each coarse neighborhood w_i is $\{\boldsymbol{\phi}_k^{w_i}\}_{k=1}^{L_i}$, and the set of partition of unity functions is $\{\chi_i\}_{i=1}^{N_v}$. Then, the following is generalized multiscale finite element formulation for Equation (2): Find $\vec{\boldsymbol{c}} = \{c_k^{w_i}\}_{i,k}$ such that

$$\int_\Omega \boldsymbol{a}(x^1, \boldsymbol{D}\boldsymbol{u}_{ms}) \cdot \boldsymbol{D}\boldsymbol{\phi}_j^{w_i}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\phi}_j^{w_i}\, dx \quad \forall\, j, \tag{15}$$

where $\boldsymbol{a}(x^1, \boldsymbol{D}\boldsymbol{u}) = \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\boldsymbol{u}|)\boldsymbol{D}\boldsymbol{u}$ was defined in Subsection 2.1.

Remark 4.1. *The numerical solution $\boldsymbol{u}_{ms}$ in (15) for the GMsFEM is uniquely defined. Indeed, suppose there exist two solutions $\boldsymbol{u}_1 = \boldsymbol{H}_2(\sum_i \sum_{k=1}^{L_i} c_{k,1}^{w_i} \chi_i \boldsymbol{\phi}_k^{w_i})$ and $\boldsymbol{u}_2 = \boldsymbol{H}_2(\sum_i \sum_{k=1}^{L_i} c_{k,2}^{w_i} \chi_i \boldsymbol{\phi}_k^{w_i})$ to (15). Consequently, it holds that*

$$\int_\Omega (\boldsymbol{a}(x^1, \boldsymbol{D}\boldsymbol{u}_1) - \boldsymbol{a}(x^1, \boldsymbol{D}\boldsymbol{u}_2)) \cdot \boldsymbol{D}\boldsymbol{v}\, dx = 0,$$

for all test functions. Given that $1 \le \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\boldsymbol{u}|)$ and that the extensions in each coarse block are $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, we obtain

$$\begin{aligned}
&\int_\Omega |\boldsymbol{D}\boldsymbol{u}_1 - \boldsymbol{D}\boldsymbol{u}_2|^2 \\
&\le \int_\Omega (\boldsymbol{\kappa}(x^1, |\boldsymbol{D}\boldsymbol{u}_1|)\boldsymbol{D}\boldsymbol{u}_1 - \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\boldsymbol{u}_2|)\boldsymbol{D}\boldsymbol{u}_2) \cdot \boldsymbol{D}(\boldsymbol{u}_1 - \boldsymbol{u}_2) \\
&= 0,
\end{aligned}$$

which implies $\boldsymbol{u}_1 = \boldsymbol{u}_2$. Therefore, the solution of (15) is uniquely defined.

4.1. Partition of unity functions

We must first build a set of partition of unity functions $\{\chi_i\}_{i=1}^{N_v}$, similarly to the suggested approach in [1]. The sum of these functions is one and they are supported in coarse neighborhoods. In particular, $\sum_{i=1}^{N_v} \chi_i = 1$, and the support of χ_i is w_i. Additionally, at the vertex $\boldsymbol{x}_i$, χ_i has a value of 1. Two widely used sets of partition of unity functions are shown below.

- First of all, it is a bilinear partition of unity: χ_i is defined as the standard bilinear basis functions χ_i^0 on w_i for each $i \in \{1, \cdots, N_v\}$:

$$\chi_i = \chi_i^0 = \begin{cases} \chi_i^0(y_i) \text{ for } y_i \in w_i\,, \\ 1 \text{ at node } \boldsymbol{x}_i\,, \\ 0 \text{ on } \partial w_i\,. \end{cases}$$

- Second, a multiscale partition of unity (with linear boundary conditions) is available for better numerical performance: With any $K \in w_i$, for some $\boldsymbol{\phi}_k^{w_i} \in \boldsymbol{H}_0^1(K)$ (to be specified later), χ_i is defined by

$$-\text{div}(\boldsymbol{a}(x^1, \boldsymbol{D}(\chi_i \boldsymbol{\phi}_k^{w_i}))) = \boldsymbol{0} \text{ in } K \in w_i\,,$$
$$\chi_i = \chi_i^0 \text{ on } \partial K\,.$$

4.2. Generalized multiscale finite element basis

4.2.1. Snapshot space

Provided a coarse neighborhood w_i, starting with a snapshot space $\boldsymbol{V}_{snap}^{w_i}$, the multiscale basis functions on w_i are constructed. The set of functions defined on w_i makes up the snapshot space $\boldsymbol{V}_{snap}^{w_i}$ and comprises all or the majority of its essential ingredients of the fine-scale solution over w_i.

In order to extract the dominant modes, which form the offline basis functions, a spectral problem is then solved in the snapshot space $\boldsymbol{V}_{snap}^{w_i}$; and offline space is the term used to describe the resulting reduced space. Listed below are two popular options for $\boldsymbol{V}_{snap}^{w_i}$.

The first option is to use all fine-grid functions that are available in w_i. Although this snapshot space provides an exact approximation of the solution space, it can also be exceedingly big. Therefore, utilizing harmonic extensions is the second option as follows.

- The collection of all nodes belonging to the fine mesh $\mathscr{T}^h$ that are located on ∂w_i is denoted by the symbol $\boldsymbol{M}_h(w_i)$.

- We build a discrete delta function $\boldsymbol{\delta}_j^h(\boldsymbol{x}_k)$ defined on $\boldsymbol{M}_h(w_i)$ for each fine-grid node $\boldsymbol{x}_k \in \boldsymbol{M}_h(w_i)$ by

$$\boldsymbol{\delta}_j^h(\boldsymbol{x}_k) = \boldsymbol{\delta}_{jk} = \begin{cases} (\delta_{jj}, 0) \text{ or } (0, \delta_{jj})\,, & k = j\,, \\ \boldsymbol{0} = (0,0)\,, & k \neq j\,. \end{cases}$$

- Then, under the notation $\boldsymbol{\psi}_j^{w_i}$, the $j-$th snapshot basis function is specified as the solution of

$$-\text{div}(\boldsymbol{\kappa}(x^1, |\boldsymbol{D}\boldsymbol{\psi}_j^{w_i}|)\boldsymbol{D}\boldsymbol{\psi}_j^{w_i}) = \boldsymbol{0} \text{ in } w_i\,,$$
$$\boldsymbol{\psi}_j^{w_i} = \boldsymbol{\delta}_j^h \text{ on } \partial w_i\,. \tag{16}$$

As before, one can choose between $\boldsymbol{\psi}_j^{w_i} = (\delta_j^h, 0)$ and $\boldsymbol{\psi}_j^{w_i} = (0, \delta_j^h)$ in 2D. The dimensions of $\boldsymbol{V}_{snap}^{w_i}$ and the size of $\boldsymbol{M}_h(w_i)$ are same.

By utilizing an auxiliary spectral decomposition with these snapshots, we build offline basis functions as shown below.

4.2.2. Offline space

The design of an appropriate nonlinear spectral problem, which will be solved in the snapshot space, serves as the foundation for the establishment of a generalized multiscale basis for solving (1) or (2) in the manner of harmonic extension. We define the following nonlinear eigenvalue problem, which can be described using the Rayleigh-Ritz method (RRM), in each coarse neighborhood w_i :

$$\begin{cases} \boldsymbol{\phi}_1^{w_i} = \boldsymbol{c}^{w_i}\,, \\ \lambda_1^{w_i} = 0\,, \\ \boldsymbol{\phi}_k^{w_i} = \arg \min\limits_{\boldsymbol{v} \in V_{snap}^{w_i}} \dfrac{G^{w_i}(\boldsymbol{v})}{G_\chi^{w_i}(\boldsymbol{v} - \boldsymbol{P}_{k-1}(\boldsymbol{v}))}\,, \\ \lambda_k^{w_i} = \dfrac{G^{w_i}(\boldsymbol{\phi}_k^{w_i})}{G_\chi^{w_i}(\boldsymbol{\phi}_k^{w_i} - \boldsymbol{P}_{k-1}(\boldsymbol{\phi}_k^{w_i}))}\,, \quad \forall k \geq 2\,, \end{cases} \tag{17}$$

where $\boldsymbol{c}^{w_i} \in \boldsymbol{V}_{snap}^{w_i}$ in w_i is a constant function,

$$G^{w_i}(\boldsymbol{v}) = \int_{w_i} \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\boldsymbol{H}_2(\boldsymbol{v})|)\, |\boldsymbol{D}\boldsymbol{H}_2(\boldsymbol{v})|^2\, dx,$$
$$G_{\chi}^{w_i}(\boldsymbol{v}) = \int_{w_i} \boldsymbol{\kappa}(x^1, |\boldsymbol{D}\boldsymbol{H}_2(\chi_i \boldsymbol{v})|)\, |\boldsymbol{D}\boldsymbol{H}_2(\chi_i \boldsymbol{v})|^2\, dx,$$
$$\boldsymbol{P}_k(\boldsymbol{u}) = \arg \min_{\boldsymbol{v} \in \boldsymbol{V}_{k-1}^{w_i}} G^{w_i}(\boldsymbol{u} - \boldsymbol{v}),$$
$$\boldsymbol{V}_{k-1}^{w_i} = \operatorname{span}\{\boldsymbol{\phi}_1^{w_i}, \cdots, \boldsymbol{\phi}_{k-1}^{w_i}\}.$$

This well-defined nonlinear eigenvalue problem is a classical orthogonal subspace minimization method (see, for instance, [8]).

After being multiplied by the corresponding partition of the unity function χ_i, the eigenfunctions $\{\boldsymbol{\phi}_k^{w_i}\}_k$ in each coarse neighborhood w_i will contribute as *offline basis* (or we refer to them as *generalized multiscale basis* or *eigenbasis*). We choose the first L_i eigenfunctions on each w_i and designate the offline space as

$$\boldsymbol{V}^c = \operatorname{span}\{\chi_i \boldsymbol{\phi}_k^{w_i} : k = 1, \cdots, L_i\,; i = 1, \cdots, N_v\} \subseteq \boldsymbol{H}^1(\Omega).$$

Keeping in mind that our solution (14) has the form $\boldsymbol{u}_{ms} = \boldsymbol{H}_2(\sum_i \sum_{k=1}^{L_i} c_k^{w_i} \chi_i \boldsymbol{\phi}_k^{w_i})$, which indicates that $\boldsymbol{u}_{ms}$ is reached by harmonically extending $\sum_i \sum_{k=1}^{L_i} c_k^{w_i} \chi_i \boldsymbol{\phi}_k^{w_i}$ in each coarse element K, and we thus only take into account the values of $\sum_i \sum_{k=1}^{L_i} c_k^{w_i} \chi_i \boldsymbol{\phi}_k^{w_i}$ restricted to each coarse edge. More precisely, in the harmonic extension process, if one coarse neighborhood w_i of an interior coarse node $\boldsymbol{x}_i$ is considered (see Figure 2 in [1], for example), then we choose the restriction of $\sum_i \sum_{k=1}^{L_i} c_k^{w_i} \chi_i \boldsymbol{\phi}_k^{w_i}$ on the related 12 coarse edges. It is important to note that the partition of unity functions χ_i become zero at and beyond the w_i's boundary. This is why the restriction of $\sum_i \sum_{k=1}^{L_i} c_k^{w_i} \chi_i \boldsymbol{\phi}_k^{w_i}$ on the *cross* (that is, the inside four coarse edges having vertex $\boldsymbol{x}_i$) is valid. These facts allow us to restrict $\chi_i \boldsymbol{\phi}_k^{w_i}$ $(k = 1, \cdots, L_i)$ on the cross of w_i and present the restricted basis (called *cross basis*) by $\hat{\boldsymbol{\phi}}_k^{w_i}$. Afterward, the following expression holds

$$\boldsymbol{u}_{ms} = \boldsymbol{H}_2\Big(\sum_i \sum_{k=1}^{L_i} c_k^{w_i} \hat{\boldsymbol{\phi}}_k^{w_i}\Big). \tag{18}$$

Let us define

$$\hat{\boldsymbol{V}}^c = \operatorname{span}\{\hat{\boldsymbol{\phi}}_k^{w_i} : k = 1, \cdots, L_i; i = 1, \cdots, N_v\}. \tag{19}$$

This allows us to concentrate on the degrees of freedom on the crosses when operating spectral decomposition.

We note that for nonlinear problems, generally, it is impossible to fully understand the effects of small separable scales without the use of nonlinear harmonic functions. In contrast, one can create a linear basis function for every coarse node that contains the small scales' effects in linear problems.

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