TẠP CHÍ KHOA HỌC & CÔNG NGHỆ ĐẠI HỌC DUY TÂN
DTU Journal of Science and Technology 5(60) (2023) 80-87

TRƯỜNG ĐẠI HỌC DUY TÂN
DUYTAN UNIVERSITY

Ulam-Hyers-Rassias stability for a random nonlinear Volterra equation

Sự ổn định Ulam-Hyers-Rassias của phương trình tích phân Volterra ngẫu nhiên

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(Ngày nhận bài: 30/3/2023, ngày phản biện xong: 05/4/2023, ngày chấp nhận đăng: 15/9/2023)

Abstract

In this paper, by using the classical Banach contraction principle, we investigate and establish stability in the sense of Ulam-Hyers and Ulam-Hyers-Rassias for random nonlinear integral equations.

Keywords: Random integral equations; Volterra equation; Banach's fixed point theorem; Ulam-Hyers-Rassias stability

Tóm tắt

Trong bài báo này, chúng tôi đưa ra định nghĩa sự ổn định Ulam-Hyers và Ulam-Hyers-Rassias cho một lớp phương trình tích phân ngẫu nhiên phi tuyến tiến dạng Volterra. Sau đó chúng tôi chứng minh rằng lớp phương trình này ổn định theo nghĩa đã định nghĩa ở trên.

Từ khóa: Phương trình tích phân ngẫu nhiên; Phương trình Volterra; Nguyên lý điểm bất động Banach; Sự ổn định Ulam-Hyers-Rassias

1. Introduction

The Ulam-Hyers-Rassias stability problem is motivated by Ulam's talk given in 1940. In the talk, he discussed a problem concerning the stability of homomorphisms. In 1941, D.H. Hyers [7] gave a partial solution to Ulam's problem. In 1978, Th.M. Rassias [11] studied a similar problem. The stability considered in [11] is often called the Ulam-Hyers-Rassias stability. The concept of the stability can also be defined for differential and integral equations, see [1, 3, 4, 5, 6, 8, 9] and the references therein. In recent years, the investigation of the stability is an active subject that has become one of the central themes of mathematical analysis.

In this paper, we first introduce the notions of Ulam-Hyers and Ulam-Hyers-Rassias stabilities for random integral equation (1) below and then prove that the equation defined on not only finite but also infinite intervals has stability in the

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senses Ulam-Hyers and Ulam-Hyers-Rassias.

$$X_t = h(t;\omega) + \int_0^t k(t,s;\omega) f(t,s,X_s) ds, t \in I, \tag{1}$$

where:

(i) $I = [0,T]$ or $I = [0,\infty)$;

(ii) $\omega \in \Omega$, where Ω is the supporting set of the probability space $(\Omega, \mathscr{F}, \mathbb{P})$;

(iii) $X_t := X(t;\omega), t \in I$, is the unknown random process;

(iv) $h(t;\omega), t \in I$, is the stochastic free term or free random variable defined for $t \in I$;

(v) the stochastic kernel $k(t,s;\omega)$ is a random variable defined for $(t,s) \in \Delta$, where $\Delta = \{(t,s) \in I^2 : 0 \le s \le t\}$;

(vi) $f(t,x)$ is a scalar function defined for $t \in I$ and $x \in \mathbb{R}$, where $\mathbb{R}$ is the real line.

This paper is organized as follows. In Section 2, the authors propose the notions of the stability and state some remarks together with Banach's fixed point theorem which will be used in proving the theorems. The authors consider the stability for the equation (1) on the finite interval in section 3 and on the infinite interval in section 4. In section 5, one example is given to illustrate some theorems of the work. One notices that the settings in papers [10] and [12] match perfectly the purpose of this paper. Moreover, we would like to stress that proving the stability of an equation defined on the infinite interval is a difficult task.

2. Preliminaries

We shall consider in (1) the random solution $X(t;\omega)$ and the stochastic free term $h(t;\omega)$ to be functions of the real argument t with values in the space $L_2(\Omega, \mathscr{F}, \mathbb{P})$. Notice here that $L_2(\Omega, \mathscr{F}, \mathbb{P})$ is a Banach space with norm $\|\cdot\|_2 = \sqrt{\mathbb{E}(\cdot)^2}$, where $\mathbb{E}$ is the expectation with respect to the probability measure $\mathbb{P}$. The random function $f(t, X_t)$, under convenient conditions, will also be a function of t with values in $L_2(\Omega, \mathscr{F}, \mathbb{P})$. The stochastic kernel $k(t,s;\omega)$ is an essentially bounded function with respect to $\mathbb{P}$ for $(t,s) \in \Delta$. It means that

$$|||k(t,s;\omega)||| := \mathbb{P} - ess\sup_{\omega} |k(t,s;\omega)| < \infty, \tag{2}$$

that is

$$|||k(t,s;\omega)||| = \inf_{\Omega_0} \left\{ \sup_{\Omega \setminus \Omega_0} |k(t,s;\omega)| \right\} < \infty, \tag{3}$$

with $\mathbb{P}(\Omega_0) = 0$. The values of the stochastic kernel for fixed t and s will be in $L_\infty(\Omega, \mathscr{F}, \mathbb{P})$, so that the product of $k(t,s;\omega)$ and $f(t,X_t)$ will always be in $L_2(\Omega, \mathscr{F}, \mathbb{P})$.

In order to show that equation (1) is stable in the senses of Ulam-Hyers and Ulam-Hyers-Rassias, we shall need some definitions and remarks in [10, 12].

Definition 2.1. *([12]) Let C_b denote the space of all continuous and bounded functions on I with values in $L_2(\Omega, \mathscr{F}, \mathbb{P})$.*

Remark 2.2. *It is known that C_b is a Banach space with norm $\|\cdot\|_{C_b}$ defined by*

$$\|X(t;\omega)\|_{C_b} = \sup_{t \in I} \|X(t;\omega)\|_2. \tag{4}$$

Definition 2.3. *([12]) Let C_ϕ denote the space of all processes $X(t;\omega) \in L_2(\Omega, \mathscr{F}, \mathbb{P})$ with $\|X(t;\omega)\|_2 \le K\phi(t), \forall t \in I$ where $\phi(t) > 0$ is a given continuous function and K is a positive constant.*

Remark 2.4. *It is known that C_ϕ is a Banach space with norm $\|\cdot\|_{C_\phi}$ defined by*

$$\|X(t;\omega)\|_{C_\phi} = \sup_{t \in I} \left\{ \frac{\|X(t;\omega)\|_2}{\phi(t)} \right\}. \tag{5}$$

Remark 2.5. *If in Definition 2.3 one has $\phi(t) = 1, \forall t \in I$, then $C_1 \equiv C_b$.*

Definition 2.6. *([10]) Let $C_{1,\phi}$ denote the space of all processes $X(t,s;\omega) \in C_b$ with $\|X(t,s;\omega)\|_2 \le K\phi(t)\phi(s), \forall 0 \le s \le t \in I$ where $\phi(t) > 0$ is a given continuous function and K is a positive constant.*

Remark 2.7. *([10]) It is known that $C_{1,\phi}$ is a Banach space with norm $\|\cdot\|_{C_{1,\phi}}$ defined by*

$$\|X(t,s;\omega)\|_{C_{1,\phi}} = \sup_{0 \le s \le t \in I} \left\{ \frac{\|X(t,s;\omega)\|_2}{\phi(t)\phi(s)} \right\}. \tag{6}$$

In the following definitions, we introduce the stability in the senses Ulam-Hyers and Ulam-Hyers-Rassias for the random integral equation.

Definition 2.8. *The equation* (1) *is said to have Ulam-Hyers stability with respect to* ϵ *if there exists a constant* $M_\epsilon > 0$ *such that for each solution* $X_t \in L_2(\Omega, \mathscr{F}, \mathbb{P})$ *of the following inequation*

$$\left\| X_t - h(t;\omega) - \int_0^t k(t,s;\omega) f(t,s,X_s)ds \right\|_2 \le \epsilon, \tag{7}$$

for all $t \in I$*, there exists a solution* $U_t \in L_2(\Omega, \mathscr{F}, \mathbb{P})$ *of the equation* (1) *such that*

$$\|X_t - U_t\|_2 \le M_\epsilon \epsilon, \forall t \in I, \tag{8}$$

where M_ϵ *is a constant that does not depend on* X_t.

Definition 2.9. *The equation* (1) *is said to have Ulam-Hyers-Rassias stability with respect to* $\phi(t)$ *if there exists a constant* $M_\phi > 0$ *such that for each solution* $X_t \in L_2(\Omega, \mathscr{F}, \mathbb{P})$ *of the following inequation*

$$\left\| X_t - h(t;\omega) - \int_0^t k(t,s;\omega) f(t,s,X_s)ds \right\|_2 \le \phi(t), \tag{9}$$

for all $t \in I$*, there exists a solution* $U_t \in L_2(\Omega, \mathscr{F}, \mathbb{P})$ *of the equation* (1) *such that*

$$\|X_t - U_t\|_2 \le M_\phi \phi(t), \forall t \in I, \tag{10}$$

where M_ϕ *is a constant that does not depend on* X_t.

For the convenience of writing in later use, we define the integral operators Γ and Λ as follows

$$\Gamma(X(t;\omega)) = \int_0^t k(t,s;\omega) f(t,s,X(s;\omega))ds, \tag{11}$$

$$\Lambda(X(t;\omega)) = h(t;\omega) + \int_0^t k(t,s;\omega) f(t,s,X(s;\omega))ds. \tag{12}$$

We now restate here the Banach's fixed point theory. This theorem will play an important role in proving our main theorems.

Theorem 2.10. *([2]) (Banach's fixed point theorem) Suppose* (X,d) *is a complete metric space and* $T: X \to X$ *is a contraction (for some* $\lambda \in [0,1)$*),* $d(T(x),T(y) \le \lambda d(x,y)$ *for all* $x,y \in X$*. Also suppose that* $u \in X, \delta > 0$*, and*

$$d(u, T(u)) \le \delta. \tag{13}$$

Then there exists a unique $p \in X$ *such that* $p = T(p)$*. Moreover,* $d(u,p) \le \dfrac{\delta}{1-\lambda}$.

3. Ulam-Hyers-Rassias stability on a finite interval

In this section, we shall show that equation (1) in which $f(t,s,X_s) \equiv f(s,X_s)$ on the finite interval $I = [0,T]$ is stable in the senses of Ulam-Hyers and Ulam-Hyers-Rassias. Furthermore, under suitable conditions, the equation also has a unique solution.

Theorem 3.1. *Suppose that the following assumptions are satisfied*

1. $h(t;\omega) \in C_b, 0 \le t \le T,$
2. $|f(t,X_t)| \le K(1+|X_t|), 0 \le t \le T,$
3. $|f(t,X_t) - f(t,Y_t)| \le \alpha |X_t - Y_t|, 0 \le t \le T,$
4. $\alpha \sup_{t\in[0,T]} \int_0^t |||k(t,s;\omega)||| ds < 1.$

Then equation (1) *has a unique solution in* C_b *and the Ulam-Hyers stability.*

Proof. For $X_t \in C_b$, using the triangle inequality, inequality $\| \int_0^t \cdot ds\|_2 \le \int_0^t \|\cdot\|_2 ds$, and the following estimation

$$\begin{aligned} &|\Lambda(X_t)| \\ &= \left| h(t;\omega) + \int_0^t k(t,s;\omega) f(s,X_s)ds \right| \\ &\le |h(t;\omega)| + \int_0^t |||k(t,s;\omega)||| |f(s,X_s)| ds. \end{aligned}$$

one gets

$$
\begin{aligned}
&\|\Lambda(X_t)\|_2 \\
&\le \|h(t;\omega)\|_2 + \left\| \int_0^t \| k(t,s;\omega) \| |f(s,X_s)| ds \right\|_2 \\
&\le \|h(t;\omega)\|_2 + \int_0^t \| k(t,s;\omega) \| \|f(s,X_s)\|_2 ds \\
&\le \|h(t;\omega)\|_2 + \int_0^t \| k(t,s;\omega) \| K(1+\|X_s\|_2) ds \\
&\le \|h(t;\omega)\|_{C_b} + K(1+\|X_s\|_{C_b}) \int_0^t \| k(t,s;\omega) \| ds \\
&\le \|h(t;\omega)\|_{C_b} \\
&\quad + K(1+\|X_s\|_{C_b}) \sup_{t\in[0,T]} \int_0^t \| k(t,s;\omega) \| ds \\
&< \infty.
\end{aligned}
$$

Hence, $\Lambda(C_b) \subset C_b$. With $X_t, Y_t \in C_b$, one gets

$$
\begin{aligned}
&|\Lambda(X_t) - \Lambda(Y_t)| \\
&= \left| \int_0^t k(t,s;\omega)(f(s,X_s) - f(s,Y_s)) ds \right| \\
&\le \int_0^t \| k(t,s;\omega) \| |f(s,X_s) - f(s,Y_s)| ds.
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|\Lambda(X_t) - \Lambda(Y_t)\|_2 \\
&\le \left\| \int_0^t \| k(t,s;\omega) \| |f(s,X_s) - f(s,Y_s)| ds \right\|_2 \\
&\le \int_0^t \| k(t,s;\omega) \| \|f(s,X_s) - f(s,Y_s)\|_2 ds \\
&\le \int_0^t \| k(t,s;\omega) \| \alpha \|X_s - Y_s\|_2 ds \\
&\le \alpha \|X_s - Y_s\|_{C_b} \int_0^t \| k(t,s;\omega) \| ds \\
&\le \alpha \|X_s - Y_s\|_{C_b} \sup_{t\in[0,T]} \int_0^t \| k(t,s;\omega) \| ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|\Lambda(X_t) - \Lambda(Y_t)\|_{C_b} \\
&\le \alpha \|X_s - Y_s\|_{C_b} \sup_{t\in[0,T]} \int_0^t \| k(t,s;\omega) \| ds.
\end{aligned}
$$

By assumption (4), the mapping Λ is strictly contractive. Thus, according to Banach's fixed point principle, equation (1) has a unique solution $U_t \in C_b$. Let $X_t \in C_b$ be a solution of the inequation (7). It means that $\|X_t - \Lambda(X_t)\|_2 \le \epsilon, \forall t \in [0,T]$, which implies $\|X_t - \Lambda(X_t)\|_{C_b} \le \epsilon$. On the one hand, by the estimation (13) in Theorem 2.10, one gets

$$\|X_t - U_t\|_{C_b} \le \frac{\epsilon}{1-C_1}, \tag{14}$$

where $C_1 = \alpha \sup_{t\in[0,T]} \int_0^t \| k(t,s;\omega) \| ds$. On the other hand, one has

$$\|X_t - U_t\|_2 \le \|X_t - U_t\|_{C_b}, \forall t \in [0,T]. \tag{15}$$

Thus, $\|X_t - U_t\|_2 \le \dfrac{\epsilon}{1-C_1}$, which implies that the equation (1) is stable in the sense Ulam-Hyers. It completes the proof.

Theorem 3.2. *Suppose that the following assumptions are satisfied*

1. $h(t;\omega) \in C_b$, $0 \le t \le T$,
2. $|f(t,X_t)| \le K(1+|X_t|)$, $0 \le t \le T$,
3. $|f(t,X_t) - f(t,Y_t)| \le \alpha |X_t - Y_t|$, $0 \le t \le T$,
4. *The function $\phi(t)$ is positive and there exists a constant $N_\phi > 0$ such that*

$$
\begin{aligned}
&\int_0^t \phi^2(s) ds \le N_\phi \phi^2(t), \forall t \in [0,T] \\
&\sup_{t\in[0,T]} \int_0^t \phi^2(s) ds < \infty,
\end{aligned}
$$

5. $\alpha \sqrt{N_\phi \sup_{t\in[0,T]} \int_0^t \left\| k^2(t,s;\omega) \right\| ds} < 1.$

Then equation (1) *has a unique solution in C_b and the Ulam-Hyers-Rassias stability with respect to $\phi(t)$.*

Proof. For all $X_t, Y_t \in C_b$, we set

$$d_\phi(X_t, Y_t) = \sup_{t\in[0,T]} \frac{\|X_t - Y_t\|_2}{\phi(t)} < \infty. \tag{16}$$

As in Theorem 3.1, one has $\Lambda(C_b) \subset C_b$ and it is known that (C_b, d_ϕ) is a complete metric space.

We assert that Λ is strictly contractive on C_b. Given any $X_t, Y_t \in C_b$, let $M_{X_t,Y_t} \in [0,\infty)$ be an arbitrary constant such that $d_\phi(X_t, Y_t) \le M_{X_t,Y_t}$, from wich we deduce that

$$\|X_t - Y_t\|_2 \le M_{X_t,Y_t}\phi(t), \quad \forall t \in [0,T]. \tag{17}$$

By Schwarz inequality, one gets

$$\begin{aligned}
&|\Lambda(X_t) - \Lambda(Y_t)|^2 \\
&= \left|\int_0^t k(t,s;\omega)(f(s,X_s) - f(s,Y_s))ds\right|^2 \\
&\le \int_0^t k^2(t,s;\omega)ds \int_0^t (f(s,X_s) - f(s,Y_s))^2 ds \\
&\le \int_0^t |||k^2(t,s;\omega)|||ds \int_0^t (f(s,X_s) - f(s,Y_s))^2 ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|\Lambda(X_t) - \Lambda(Y_t)\|_2^2 \\
&\le \left\|\int_0^t |||k^2(t,s;\omega)|||ds \int_0^t (f(s,X_s) - f(s,Y_s))^2 ds\right\|_2 \\
&= \int_0^t |||k^2(t,s;\omega)|||ds \left\|\int_0^t (f(s,X_s) - f(s,Y_s))^2 ds\right\|_2 \\
&\le \int_0^t |||k^2(t,s;\omega)|||ds \int_0^t \|(f(s,X_s) - f(s,Y_s))\|_2^2 ds \\
&\le \int_0^t |||k^2(t,s;\omega)|||ds \int_0^t \alpha^2 \|X_s - Y_s\|_2^2 ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|\Lambda(X_t) - \Lambda(Y_t)\|_2^2 \\
&\le \alpha^2 \int_0^t |||k^2(t,s;\omega)|||ds \cdot \int_0^t M^2_{X_t,Y_t}\phi^2(s)ds \\
&\le \alpha^2 \int_0^t |||k^2(t,s;\omega)|||ds \cdot M^2_{X_t,Y_t} N_\phi \phi^2(t).
\end{aligned}$$

Hence,

$$\|\Lambda(X_t) - \Lambda(Y_t)\|_2 \le C_2 M_{X_t,Y_t}\phi(t) \tag{18}$$

where $C_2 = \alpha\sqrt{N_\phi \sup_{t\in[0,T]} \int_0^t |||k^2(t,s;\omega)|||ds}$. It implies that $d_\phi(\Lambda(X_t), \Lambda(Y_t)) \le C_2 M_{X_t,Y_t}$. Thus, one concludes that $d_\phi(\Lambda(X_t), \Lambda(Y_t)) \le C_2 d_\phi(X_t, Y_t)$ for any $X_t, Y_t \in C_b$. By assumption (5), the mapping Λ is strictly contractive on the metric space (C_b, d_ϕ). Thus, by the Banach's fixed point principle, equation (1) has a unique solution in the space C_b.

Let X_t be a solution of the inequation (7) and let U_t be the solution of the equation (1). Since $\|X_t - \Lambda(X_t)\|_2 \le \phi(t), \ \forall t \in [0,T]$, one has $d_\phi(X_t, \Lambda(X_t)) \le 1$. By the triangle inequality, one obtains

$$\begin{aligned}
d_\phi(X_t, U_t) &\le d_\phi(X_t, \Lambda(X_t)) + d_\phi(\Lambda(X_t), U_t) \\
&\le d_\phi(X_t, \Lambda(X_t)) + d_\phi(\Lambda(X_t), \Lambda(U_t)) \\
&\le 1 + C_2 d_\phi(X_t, U_t),
\end{aligned}$$

which implies that

$$d_\phi(X_t, U_t) \le \frac{1}{1 - C_2}. \tag{19}$$

Hence,

$$\|X_t - U_t\|_2 \le C_\phi \phi(t), \tag{20}$$

where $C_\phi = \dfrac{1}{1 - C_2}$. It means that equation (1) has the Ulam-Hyers-Rassias stability. The proof of the theorem thus is complete.

4. Ulam-Hyers-Rassias stability on an infinite interval

In this section, by making use of some similar assumptions on equation (1) given in the papers [10] and [12], we shall prove that the equation defined on the infinite interval $I = [0, \infty)$ is stable in the senses of Ulam-Hyers and Ulam-Hyers-Rassias.

Theorem 4.1. *Consider the random integral equation* (1) *under the following conditions*

1. $h(t;\omega) \in C_b, 0 \le t,$
2. $|f(t,s,X_s)| \le \gamma(t,s)|X_s|, 0 \le s \le t,$ *where* $\gamma(t,s)$ *is a positive function.*
3. $|f(t,s,X_s) - f(t,s,Y_s)| \le \alpha\gamma(t,s)|X_s - Y_s|, 0 \le s \le t,$
4. $\alpha \sup_{t\ge 0} \int_0^t \gamma(t,s)|||k(t,s;\omega)|||ds < 1.$

Then equation (1) *has a unique solution in* C_b *and the Ulam-Hyers stability.*

Proof. As in Theorem 3.1, with $X_t \in C_b$, one gets

$$\begin{aligned}
&\|\Lambda(X_t)\|_2 \\
&\le \|h(t;\omega)\|_2 + \int_0^t |||k(t,s;\omega)||| \, \|f(t,s,X_s)\|_2 ds \\
&\le \|h(t;\omega)\|_2 + \int_0^t |||k(t,s;\omega)|||\gamma(t,s)\|X_s\|_2 ds \\
&\le \|h(t;\omega)\|_2 + \|X_s\|_{C_b} \int_0^t |||k(t,s;\omega)|||\gamma(t,s) ds \\
&\le \|h(t;\omega)\|_{C_b} + \|X_s\|_{C_b} \sup_{t\ge 0} \int_0^t |||k(t,s;\omega)|||\gamma(t,s) ds \\
&< \infty.
\end{aligned}$$

Therefore, $\Lambda(C_b) \subset C_b$.

With $X_t, Y_t \in C_b$, one gets

$$\begin{aligned}
&\|\Lambda(X_t) - \Lambda(Y_t)\|_2 \\
&\le \int_0^t |||k(t,s;\omega)||| \, \|f(t,s,X_s) - f(t,s,Y_s)\|_2 ds \\
&\le \int_0^t |||k(t,s;\omega)|||\alpha\gamma(t,s)\|X_s - Y_s\|_2 ds \\
&\le \alpha \|X_s - Y_s\|_{C_b} \int_0^t |||k(t,s;\omega)|||\gamma(t,s) ds \\
&\le \alpha \|X_s - Y_s\|_{C_b} \sup_{t\ge 0} \int_0^t |||k(t,s;\omega)|||\gamma(t,s) ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|\Lambda(X_t) - \Lambda(Y_t)\|_{C_b} \\
&\le \|X_s - Y_s\|_{C_b} \alpha \sup_{t\ge 0} \int_0^t \gamma(t,s)|||k(t,s;\omega)||| ds.
\end{aligned}$$

By assumption (4), we state that Λ is strictly contractive. Thus, by Banach's fixed point principle, equation (1) has a unique solution in C_b.

Let X_t be a solution of inequation (7) and let U_t be the solution of equation (1). As in Theorem 3.1, one obtains

$$\|X_t - U_t\|_2 \le \frac{\epsilon}{1 - C_3}, \forall t \in [0,\infty), \tag{21}$$

where $C_3 = \alpha \sup_{t\ge 0} \int_0^t \gamma(t,s)|||k(t,s;\omega)||| ds$, which implies that equation (1) is stable in the sense Ulam-Hyers and completes the proof.

Theorem 4.2. *Consider the random integral equation* (1) *under the following conditions*

1. $h(t;\omega) \in C_\phi$,
2. $|f(t,s,X_s)| \le \phi(t)[z(t,\omega) + \gamma(t,s)|X_s|]$ *for* $0 \le s \le t < \infty$, *where* $z(s,\omega)$ *is a second order stochastic process in* C_ϕ *and* $\gamma(t,s) > 0$ *is a bounded continuous function defined for* $0 \le s \le t$,
3. $|f(t,s,X_s) - f(t,s,Y_s)| \le \alpha\phi(t)|X_s - Y_s|, 0 \le s \le t$,
4. $\alpha \sup_{t\ge 0} \int_0^t |||k(t,s;\omega)|||\phi(s) ds < 1$.

Then equation (1) *has a unique solution in* C_ϕ *and the Ulam-Hyers-Rassias stability.*

Proof. According to [10], $(C_{1,\phi}, C_\phi)$ is admissible with respect to operator Γ in (11). That is $\Gamma(C_{1,\phi}) \subset C_\phi$. Condition (2) implies that $k(t,s;\omega)f(t,s,X_s)$ is in $C_{1,\phi}$ whenever $X_t \in C_\phi$. Therefore, $\Lambda(C_\phi) \subset C_\phi$.

We shall show that if $X_t, Y_t \in C_\phi$ then $k(t,s;\omega)f(t,s,X_s) - k(t,s;\omega)f(t,s,Y_s)$ belongs to $C_{1,\phi}$. By assumption (3), one has

$$\begin{aligned}
&\frac{|k(t,s;\omega)f(t,s,X_s) - k(t,s;\omega)f(t,s,Y_s)|}{\phi(s)\phi(t)} \\
&\le \alpha |||k(t,s;\omega)||| \frac{|X_s - Y_s|}{\phi(s)},
\end{aligned}$$

which implies that $\|k(t,s;\omega)f(t,s,X_s) - k(t,s;\omega)f(t,s,Y_s)\|_{C_{1,\phi}} \le \alpha \sup_{0\le s\le t} |||k(t,s;\omega)||| \, \|X_s - Y_s\|_{C_\phi} < \infty$. Hence, $k(t,s;\omega)f(t,s,X_s) - k(t,s;\omega)f(t,s,Y_s) \in C_{1,\phi}$ when $X_s, Y_s \in C_\phi$. Again, since $(C_{1,\phi}, C_\phi)$ is admissible with respect to Γ, one has $\int_0^t k(t,s;\omega)f(t,s,X_s) - k(t,s;\omega)f(t,s,Y_s) ds \in C_\phi$. Thus, one has $\Lambda(C_\phi) \subset C_\phi$.

As in proof of Theorem 4.1, one has the following estimates

$$\begin{aligned}
&\|\Lambda(X_t) - \Lambda(Y_t)\|_2 \\
&\le \int_0^t |||k(t,s;\omega)||| \, \|f(t,s,X_s) - f(t,s,Y_s)\|_2 ds \\
&\le \alpha\phi(t) \int_0^t |||k(t,s;\omega)||| \, \|X_s - Y_s\|_2 ds.
\end{aligned}$$

Therefore,

$$\frac{\|\Lambda(X_t)-\Lambda(Y_t)\|_2}{\phi(t)} \le \alpha\int_0^t |||k(t,s;\omega)|||\,\|X_s-Y_s\|_2 ds \le \alpha\int_0^t |||k(t,s;\omega)|||\frac{\|X_s-Y_s\|_2}{\phi(s)}\phi(s)ds \le \alpha\sup_{t\ge0}\left(\int_0^t |||k(t,s;\omega)|||\phi(s)ds\right)\|X_s-Y_s\|_{C_\phi}.$$

from which we deduce that

$$\|\Lambda(X_t)-\Lambda(Y_t)\|_{C_\phi} \le \alpha\sup_{t\ge0}\left(\int_0^t |||k(t,s;\omega)|||\phi(s)ds\right)\|X_s-Y_s\|_{C_\phi}. \tag{22}$$

By assumption (4), the mapping Λ is strictly contractive. Thus, according to Banach's fixed point principle, equation (1) has a unique solution $U_t \in C_\phi$.

Let $X_t \in C_\phi$ be a solution of inequation (9). We have

$$\|X_t-\Lambda(X_t)\|_2 \le \phi(t), \tag{23}$$

from which, we deduce the following inequality $\|X_t-\Lambda(X_t)\|_{C_\phi} \le 1$.

By the triangle inequality, we get:

$$\|X_t-U_t\|_{C_\phi} \le \|X_t-\Lambda(X_t)\|_{C_\phi}+\|\Lambda(X_t)-\Lambda(U_t)\|_{C_\phi} \le 1+C_4\|X_t-Y_t\|_{C_\phi},$$

where $C_4 = \alpha\sup_{t\ge0}\int_0^t |||k(t,s;\omega)|||\phi(s)ds$. Therefore,

$$\|X_t-U_t\|_{C_\phi} \le \frac{1}{1-C_4}. \tag{24}$$

Thus, $\|X_t-U_t\|_2 \le \frac{1}{1-C_4}\phi(t), \forall t\ge 0$, which implies that equation (1) has the Ulam-Hyers-Rassias stability with respect to $\phi(t)$. This ends the proof. □

5. An example

Let us consider the following random equation on probability space $(\Omega,\mathscr{F},\mathbb{P})$ that is one dimensional case in [12]

$$\dot{X}(t,\omega) = A(\omega)X(t;\omega)+f(t,X(t;\omega)), \tag{25}$$

for $t \ge 0$ and where

1. $X(t;\omega)$ is the unknown random process,
2. $A(\omega)$ is a random variable on the probability space,
3. $f(t,x)$ is a given function for $t \ge 0$ and $x\in\mathbb{R}$ such that

$$|f(t,x)-f(t,y)| \le \alpha|x-y|$$

with $f(t,0)=0$ and α sufficiently small.

The above random equation can be reduced to the following random integral equation

$$X(t;\omega) = e^{A(\omega)t}X_0(\omega)+\int_0^t e^{A(\omega)(t-s)}f(s,X(s;\omega))ds \tag{26}$$

where $X_0(\omega) := X(0,\omega)$. Let the free stochastic term of our equation be

$$h(t;\omega) = e^{A(\omega)t}X_0(\omega)$$

and let the stochastic kernel be

$$k(t,s;\omega) = e^{A(\omega)(t-s)},\ \ 0\le s\le t<\infty.$$

As shown in Section 4 of [12], the following holds for $\omega\in\Omega_0$

$$|||k(t,s;\omega)||| \le Ke^{-\gamma(t-s)}, \tag{27}$$

where Ω_0 is a subset of Ω such that $\mathbb{P}(\Omega_0)=1$, $K>0$ and α as defined above. For the sake of simplicity, we suppose that $A(\omega)$ is almost sure bounded above by $-\gamma$ where γ is a positive number. Thus, one has

$$|||k(t,s;\omega)||| \le e^{-\gamma(t-s)} \tag{28}$$

which implies

$$\int_0^t |||k(t,s;\omega)|||ds \le \int_0^t e^{-\gamma(t-s)}ds = \frac{1}{\gamma}(1-e^{-\gamma t}) < \frac{1}{\gamma}.$$

If $\frac{\alpha}{\gamma} < 1$ and under bounded condition on random variable $X_0(\omega)$, all conditions in Theorem 3.1 are satisfied. One can conclude that the equation (25) is stable in the sense of Ulam-Hyers. Similarly, notice in this case that the function $\gamma(t,s)$ is 1. One can show all conditions in Theorem 4.1 are satisfied as well.

Furthermore, for the case $I = [0,1]$, if one chooses the function $\phi(t) = t$, then the constant N_ϕ in Theorem 3.2 is 1/3. So if $\frac{\alpha}{\sqrt{6\gamma}} < 1$, all conditions in Theorem 3.2 are satisfied. Thus, equation (25) is stable in the sense of Ulam-Hyers-Rassias with respect to ϕ.

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